

Stability of active transmission-line metamaterials

A. D. Boardman^{*a}, Y. G. Rapoport^{a,b}, V. N. Malnev^b and R. C. Mitchell-Thomas^a.

^aJoule Physics Laboratory, Institute for Materials Research, University of Salford.

^bPhysics Faculty, Taras Shevchenko Kyiv National University,

Prospect Glushkov 6, 22 Kyiv, Ukraine

ABSTRACT

The question of whether stable, active metamaterials can be created is addressed, both through a discussion of absolute instability and an analysis of a transmission-line that produces dispersion analogous to that of the familiar split-ring resonator/wire-based metamaterial. Gain is introduced using negative conductance diodes, and it is shown that the frequency bandwidth controls the window of stable gain. The diodes are located as lumped elements in the unit cell. It is demonstrated that the production of a stable, active, negative phase frequency window is possible.

Keywords: Active media, transmission-line, metamaterial.

1. INTRODUCTION

There is a problem associated with the implementation of a meta-device that is based upon multi-dimensional metamaterials that have very high intrinsic losses. It has been known for a number of years that although the metamaterial properties are very attractive, they are usually associated with steep resonances, and this means a deep association with high losses. It is necessary, therefore, to address this problem and one way to do this is to include a special diode that is located within a meta-particle in such a way that full advantage of negative resistance can be taken. It has already been shown, however, that one of the problems of introducing gain is the possibility of introducing absolute instability [1,2]. The problem associated with the latter, are well known in plasma physics [2], and it emerges that some general rules can be formulated that permits the definition of a loss free, or amplifying frequency window. These are examined in detail in this paper using a straight forward transmission-line model that has previously been extensively discussed [3]. From transmission-line theory, it soon becomes clear that a loss free, negative phase, system can be devised. However, here loss and gain are deliberately introduced in order to provide an interesting analogue of how a split-ring resonator/metal wire metamaterial would behave if an attempt to control or reduce the loss was made. It will be shown that there is, indeed, a frequency window that supports stable amplification. This outcome directly contradicts a recent assertion [4], that any attempt to address losses in a negative phase metamaterial would result in the disappearance of the negative phase behavior. An interesting development is that if the active element has a finite frequency bandwidth, then this is an important control property that leads, in the application considered here, to stability.

2. FORMULATION OF THE PROBLEM AND ANALYTICAL RESULTS

The electromagnetic wave propagating along the transmission-line shown in fig 1, have a space time dependence of the form $\exp[i(\omega t - kx)]$, where ω and k are frequency and wave number, respectively, and it is assumed that the transmission-line extends along the x-axis. Fig 1 shows that the inductance L_r , capacitance C_r , and resistance R_r , that define the metaparticle (split-ring), are in parallel and, without showing any detail here, this means that the assumed dependence of the permeability is of Lorentz form [3]. The diode in the unit cell is also attached in parallel, and the general form of Fig. 1 is what is known in transmission-line theory as a T-configuration.

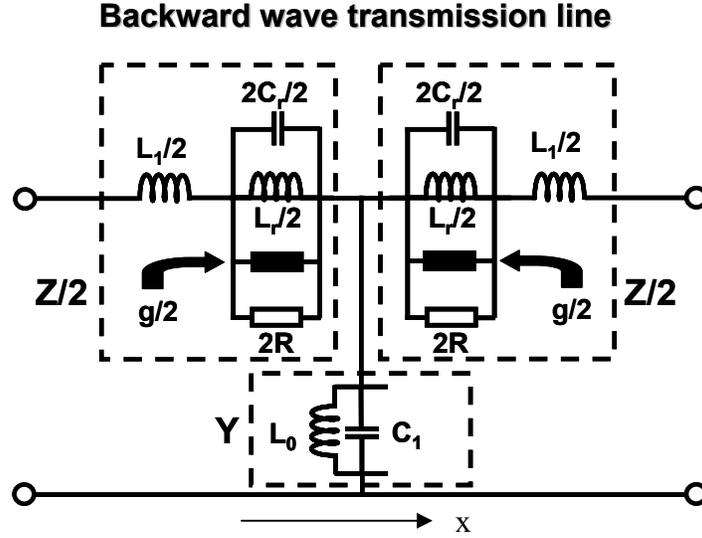


Fig. 1. Equivalent circuit of the elementary cell of a transmission line.

A novel feature is to introduce the frequency dependence of the diode conductance and to assume, that the diode is operating in a negative conductance regime. The form assumed here for the frequency dependence conductance is

$$g = g(\omega) = g_0 / [1 + i(\frac{\omega^2 - \omega_0^2}{\omega \Delta \omega})] \quad (1)$$

The structure of g actually corresponds to some type of Gunn diode [5].

Here g_0 is a constant and $g_0 < 0$, ω_0 is the frequency where the module of negative diode resistance reaches a maximum and $\Delta\omega$ is the effective bandwidth. If d is the effective width of the unit cell shown in Fig. 1, then the dispersion equation for waves propagating along the transmission line is

$$\cos(kd) = 1 + YZ/2, \text{ or } F(\omega, k) = \cos(kd) - 1 - YZ/2 = 0 \quad (2)$$

where, $Z/2$ is the impedance that describes each symmetrical horizontal part of the T- circuit. Clearly, Y describes the vertical part of T-circuit. In the so- called long-wave approximation, $kd \ll 1$, the dispersion equation reduces to

$$(kd)^2 = - YZ \quad (3)$$

The emphasis now, is to find the conditions under which absolute instability could occur [2]. In fact, sources of possible instability could be connected with either **(I)** roots or **(II)** poles of dispersion equation. Examining these in turn leads to the following conclusions.

(I) Roots

In this case it is the roots of the system of equations

$$F(\omega, k) = 0 \text{ and } \partial F / \partial k = 0 \quad (4)$$

Using the relations in (4), two possible conditions are obtained

$$kd = 0 : \quad Y(\omega)Z(\omega) = 0 \quad (5)$$

$$kd = -\pi : \quad Y(\omega)Z(\omega) = -4 \quad (6)$$

(II) Poles

Again, using equation (2), the poles of the appropriate quantities can readily be found. It is however a laborious process and the details will not be given here.

The outcome of the search for absolute instability shows that absolute instability can be avoided if a positive imaginary part to the roots determined by the conditions (I) and (II) exists. A full analysis of the roots and poles is quite laborious, but it is interesting that the condition for the absence of absolute instability, obtained from relation (5) and consideration of poles, implies an elegant restriction on g_0 given by

$$1/R_r < (-g_0) < (1+S)(1/R_r) \quad (7)$$

S is determined by the particular values associated with the circuit in the Fig. 1. Also, $S \rightarrow 0$, when $\Delta\omega/\omega_0 \rightarrow \infty$. Clearly then, a finite effective frequency bandwidth is necessary to create a practical condition for the absence of absolute instability. It can also be shown, that for the parameters chosen for the numerical calculations, no new instability points arise from the analysis of relation (6).

3. RESULTS OF THE NUMERICAL CALCULATIONS

The analysis outlined in section 2 can now be adopted to produce some numerical confirmation of stable, amplifying, waves that are free of any possibility of absolute instability. The purpose of avoiding absolute instability is to prevent the introduction of gain providing an opportunity for noise to build up everywhere in time, and hence destroy the operation of any possible device. Naturally, it is still of interest to access amplifying waves using the data that places the material safely in the non-absolute instability regime. In this section, therefore, the data characterizing the stable regime is adopted, the wave number is set equal to $k = k' + k''$ and the frequency is assumed to be real. The dispersion equations are solved under these conditions and figures 2 and 3 illustrate how the real and imaginary parts of the wave number behave with frequency.

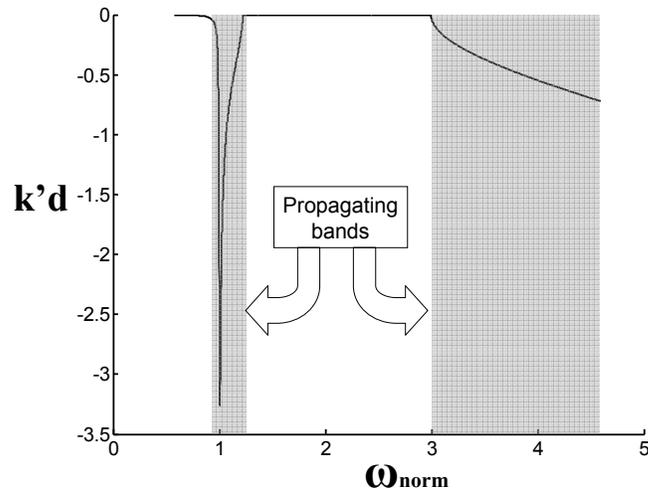


Fig. 2. Dependence of k' , the real part of wave number propagating along the transmission-line of Fig. 1, and derived from the approximate dispersion equation. $R=3.34 \cdot 10^4 \Omega$, $g_0 = -4.24 \cdot 10^{-4} \Omega^{-1}$. The non-propagating regimes are characterized by $|k''| > |k'|$ and within the scale of the figure, $|k'|$ is practically zero. Apart from the data specifically shown in the caption, the data that already exists for transmission-line models in the literature has been deployed. The frequency, ω_{norm} , is normalized with respect to the resonant frequency, $\omega_r = 1 / \sqrt{C_r L_r}$, and wave number is normalized with respect to the period of the structure, d .

Fig. 2 shows that there are some useful propagation bands. Numerical calculation show that there is essentially a practice band gap, where $|k''| > |k'|$. Apart from the data specifically shown in the caption, the data that already exists for transmission-line models in the literature has been deployed.

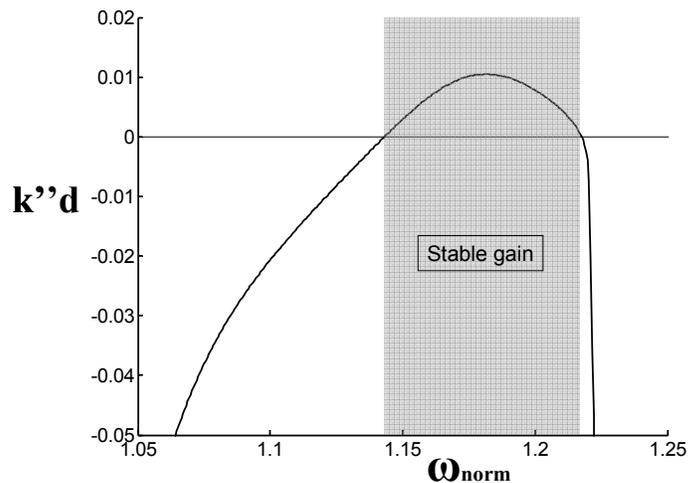


Fig. 3 Dependence of the normalized imaginary part of the wave number $k''d$, upon normalized frequency, ω_{norm} . The frequency units are the same as in Fig. 2. $R=3.34*10^4 \Omega$, $G_0=-4.24*10^{-4} \Omega^{-1}$. The relative bandwidth of the amplification region is of order of a few percent. Note that the variables are normalized with respect to the resonant frequency.

Fig. 3 picks out, specifically, a stable frequency regime that shows that k'' can go above zero. In fact, a clear frequency regime exists over which gain, defined as $k''>0$ exists. This conclusion is very interesting, and does contradict a recent assertion that this is impossible [4]. Apart from the data shown in the caption, the rest of the data is once again drawn from the literature.

4. CONCLUSIONS

The question of how to control loss in a metamaterial is addressed in the above using a transmission-line analogue. The whole question of whether the addition of gain leads to absolute instability is carefully examined and a possible frequency window with stable gain is predicted using diodes that display negative resistance and a clearly assessable conductance bandwidth. The transmission-line is developed using what is known as the T-formulation, and a gain controlled lossy resonant circuit is added into each unit cell. The detailed analysis leads to an elegant inequality that the diode conductance must satisfy, and this leads to the identification of a well defined amplification region that is potentially useful in metamaterial devices.

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