Integral Transforms for Engineers

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Preface to the 1999 Printing

The use of Fourier integrals in mathematics and physics applications dates back to the pioneering work of Joseph Fourier (1768–1830). Since that time, the notion of the integral transform has emerged as a related tool that owes much of its success to the work of Oliver Heaviside (1850–1925), an English electrical engineer who popularized the use of operational methods in differential equations and electrical engineering. During the last decade or so there have been significant generalizations of the idea of integral transforms and many new uses of the transform method in engineering and physics applications. Some of these new applications have prompted the development of very specialized transforms, such as the wavelet transform, that have their roots, however, deeply entrenched in the classical theory of Fourier. As a result, knowledge of the properties and use of classical integral transforms, such as the Fourier transform and Laplace transform, are just as important today as they have been for the last century or so.

This text was written in 1988 as an introductory treatment of integral transforms for practicing engineers and scientists, including the Fourier, Laplace, Mellin, Hankel, finite, and discrete transforms. Like the fate of many modern textbooks, the original publishing company changed hands and this book went out of print after a few years. Nonetheless, a number of individuals took the time to let us know they found the book useful as either a personal reference text or as a classroom text, and also expressed their disappointment in seeing it go out of print. We are therefore grateful to the SPIE PRESS for agreeing to bring the book back into print. As authors, we have taken this opportunity to correct several typographical errors that appeared in the first printing, but would welcome hearing from anyone who finds additional typographical errors that we did not catch or who cares to give any suggestions for further improvements as well.

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Preface

IN RECENT YEARS, INTEGRAL TRANSFORMS have become essential working tools of every engineer and applied scientist. The Laplace transform, which undoubtedly is the most familiar example, is basic to the solution of initial value problems. The Fourier transform, while being suited to solving boundary-value problems, is basic to the frequency spectrum analysis of time-varying waveforms. The purpose of this text is to introduce the use of integral transforms in obtaining solutions to problems governed by ordinary and partial differential equations and certain types of integral equations. Some other applications are also covered where appropriate.

The Laplace and Fourier transforms are by far the most widely used of all integral transforms. For this reason they have been given a more extensive treatment in this book than other integral transforms. However, there are several other integral transforms that also have been used successfully in the solution of certain boundary-value problems and in other applications. Included in this category are Mellin, Hankel, finite, and discrete transforms, which have also been given some discussion here.

The text is directed primarily toward senior and beginning graduate students in engineering sciences, physics, and mathematics who desire a deeper knowledge of transform methods than can be obtained in introductory courses in differential equations and other similar courses. It can also be used as a self-study text for practicing engineers and applied scientists who wish to learn more about the general theory and use of integral transforms. We assume the reader has a basic knowledge of differential equations and contour integration techniques from complex variables. However, most of the material involving complex variables occurs in separate sections so that much of the text can be accessible to those with a minimum background in complex variable methods. As an aid in this regard, we have included a brief appendix relevant to our use of the basic concepts and theory of complex variables in the text. Also, because of the close association of special functions and integral transforms, the first chapter is a short introduction to several of the special functions that arise quite frequently in applications. This is considered an optional chapter for those with some acquaintance with these functions, and thus it is possible to start the text with Chap. 2. Most chapters are independent of one another so that various arrangements of the material are possible.

Applications occur throughout the text and are drawn from the fields of mechanical vibration, heat conduction, potential theory, mechanics of solids and fluids, probability and statistics, and several other areas. A working knowledge in any of these areas is generally sufficient to work the examples and exercises.

In our treatment of integral transforms we have excised formal proofs in several places, but then usually make an appropriate reference for the more formal aspects of the theory. In the applications we often make the assumptions as to the commutability of certain limiting operations, and the derivation of a particular solution sometimes may not be rigorous. However, the approach adopted here is adequate in the usual applications in engineering and applied sciences. We have included a large number of worked examples and exercises to illustrate the versatility and adequacy of this approach in applications to physical problems.

We wish to thank Jack Repcheck, Senior Editor of Scientific and Technical Books department at Macmillan, for his assistance in getting this text published in a timely manner. We also wish to express our appreciation to the production staff of Macmillan for their fine efforts. Finally, we wish to acknowledge Martin Otte who corrected several errors during a final reading of the manuscript.

Introduction

The classical methods of solution of initial and boundary value problems in physics and engineering sciences have their roots in Fourier's pioneering work. An alternative approach through integral transform methods emerged primarily through Heaviside's efforts on operational techniques. In addition to being of great theoretical interest to mathematicians, integral transform methods have been found to provide easy and effective ways of solving a variety of problems arising in engineering and physical science. The use of an integral transform is somewhat analogous to that of logarithms. That is, a problem involving multiplication or division can be reduced to one involving the simpler processes of addition or subtraction by taking logarithms. After the solution has been obtained in the logarithm domain, the original solution can be recovered by finding an antilogarithm. In the same way, a problem involving derivatives can be reduced to a simpler problem involving only multiplication by polynomials in the transform variable by taking an integral transform, solving the problem in the transform domain, and then finding an inverse transform. Integral transforms arise in a natural way through the principle of linear superposition in constructing integral representations of solutions of linear differential equations.

By an *integral transform*, we mean a relation of the form*

$$\int_{-\infty}^{\infty} K(s,t)f(t)\,dt = F(s) \tag{0.1}$$

such that a given function $f(t)$ is transformed into another function $F(s)$ by means of an integral. The new function $F(s)$ is said to be the *transform* of $f(t)$, and $K(s,t)$ is called the *kernel* of the transformation. Both $K(s, t)$ and $f(t)$ must satisfy certain conditions to ensure existence of the integral and a unique transform function $F(s)$. Also, generally speaking, not more than one function $f(t)$ should yield the same transform $F(s)$. When both of the limits of integration in the defining integral are finite, we have what is called a *finite transform*.

Within the above guidelines there are a variety of kernels that may be used to define particular integral transforms for a wide class of functions $f(t)$. If the kernel is defined by

$$K(s,t) = \begin{cases} 0, & t < 0 \\ e^{-st}, & t \geq 0 \end{cases} \tag{0.2}$$

the resulting transform

$$\int_{0}^{\infty} e^{-st} f(t)\,dt = F(s) \tag{0.3}$$

is called the *Laplace transform*. When

$$K(s,t) = \frac{1}{\sqrt{2\pi}} e^{ist} \tag{0.4}$$

we generate the *Fourier transform*†

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ist} f(t)\,dt = F(s) \tag{0.5}$$

which, when t is restricted to the positive real line, leads to the *Fourier sine* and *Fourier cosine transforms*

$$\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(t) \sin st\,dt = F(s) \tag{0.6}$$

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* We will always interpret integrals like (0.1) as the *principal value* of the integral, defined in general by $PV \int_{-\infty}^{\infty} f(x)dx = \lim_{R \to \infty} \int_{-R}^{R} f(x)dx$.

† Other definitions of $K(s,t)$ for Fourier transforms involve the choices e^{ist}, e^{-ist}, $(1/2\pi)e^{ist}$, among others.

and

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\,dt = F(s) \tag{0.7}$$

The Laplace and Fourier transforms are by far the most prominent in applications. Many other transforms have been developed, but most have limited applicability. In addition to the Laplace and Fourier transforms, the next most useful transforms are perhaps the *Hankel transform of order ν*

$$\int_0^\infty tJ_\nu(st)f(t)\,dt = F(s) \tag{0.8}$$

where $J_\nu(x)$ is the Bessel function of the first kind (see Sec. 1.4), and the *Mellin transform*

$$\int_0^\infty t^{s-1}f(t)\,dt = F(s) \tag{0.9}$$

The Hankel transform arises naturally in solving boundary value problems formulated in cylindrical coordinates while the Mellin transform is useful in the solution of certain potential problems formulated in wedge-shaped regions.

The integral transforms mentioned thus far are applicable to problems involving either semiinfinite or infinite domains. However, in applying the method of integral transforms to problems formulated on finite domains it is necessary to introduce finite intervals on the transform integral. Transforms of this nature are called *finite integral transforms*.

A basic problem in the use of integral transforms is to determine the function $f(t)$ when its transform $F(s)$ is known. We refer to this as the *inverse problem*. In many cases the solution of the inverse problem is another integral transform relation of the type

$$\int_D H(s,t)F(s)\,ds = f(t) \tag{0.10}$$

where $H(s,t)$ is another kernel and D is the domain of s. Such a result is called an *inversion formula* for the particular transform. For example, the inversion formula for the Fourier transform takes the form (see Sec. 2.4)

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-ist}F(s)\,ds = f(t) \tag{0.11}$$

which is very much like the transform itself in Eq. (0.5). This means that the problems of evaluating transforms or inverse transforms are essentially the same for Fourier transforms. This is not necessarily the case for other transforms like the Laplace transform, however, where the inversion formula is quite distinct from that of the transform integral. Also, in the case of finite transforms, the inverse transform is in the form of an infinite series.

The basic aim of the transform method is to transform a given problem into one that is easier to solve. In the case of an ordinary differential equation with constant coefficients, the transformed problem is algebraic. The effect of applying an integral transform to a partial differential equation is to reduce it to a partial differential equation in one less variable. The solution of the transformed problem in either case will be a function of the transformed variable and any remaining independent variables. Inversion of this solution produces the solution of the original problem.

The exponential Fourier transform does not incorporate any boundary conditions in transforming the derivatives. Thus, it is best suited for solving differential equations on infinite domains where the boundary conditions usually only require bounded solutions. On the other hand, the Fourier cosine and sine transforms are well suited for solving certain problems on semiinfinite domains where the governing differential equation involves only even-order derivatives. We will see that the Fourier transform lends itself nicely to solving boundary-value problems associated with the following partial differential equations:

(a) the *heat equation:*

$$\nabla^2 u = a^{-2}u_t - q(x,y,z,t) \tag{0.12}$$

(b) the *wave equation:*

$$\nabla^2 u = c^{-2}u_{tt} - q(x,y,z,t) \tag{0.13}$$

(c) the *potential equation:*

$$\nabla^2 u = 0 \tag{0.14}$$

In addition, it is useful in the solution of linear integral equations of the form

$$f(x) = u(x) - \lambda \int_{-\infty}^{\infty} k(x,t)u(t)\,dt \tag{0.15}$$

and certain ordinary differential equations. Interesting applications of these transform methods arise in hydrodynamics, heat conduction, potential theory, and elasticity theory, among other areas. The Fourier transform also lends itself to the theory of probability and statistics. For example, it turns out that the moments of a random variable X are merely the coefficients of $(it)^k/k!$ in the Maclaurin series expansion of the characteristic function $C(t)$ of the random variable X, and this function is related to the probability density function $p(x)$ by the Fourier transform relation

$$C(t) = \int_{-\infty}^{\infty} e^{itx}\, p(x)\, dx \tag{0.16}$$

While the Fourier transform is suited for boundary-value problems, the Laplace transform is suited for initial-value problems. However, there are other situations for which the Laplace transform can also be used, such as in the evaluation of certain integrals and in the solution of certain integral equations of convolution type like

$$\int_0^t u(\tau)k(t - \tau)\, d\tau = f(t), \qquad t > 0 \tag{0.17}$$

In addition to the transforms mentioned above, there are other less well known transforms like the *Hilbert transform* and the *Sturm–Liouville transform*, both of which are more limited in their usefulness than the Fourier and Laplace transforms. Also, discrete transforms like the *discrete Fourier transform* (which is the discrete analog of the Fourier transform) and the *Z transform* (which is the discrete analog of the Laplace transform) are becoming more prominent in various engineering applications where it is either impossible or inconvenient to use more conventional transforms.

Much of our initial discussion will evolve around the problem of calculating the transforms $F(s)$ of given functions $f(t)$, and also around the related problem of finding inverse transforms of various functions $F(s)$. Our primary objective is to introduce methods to use the integral transforms, rather than concerning ourselves too deeply with the general theory itself. Therefore, we do not attempt to present the basic theorems in their most general forms. However, the conditions put forth in the theorems are generally broad enough to embrace most of the functions that naturally arise in engineering and physical situations. Proofs of the theorems are provided when feasible, but are sometimes based on heuristic arguments instead of rigorous mathematical procedures. For example, often we have the need in our proofs for interchanging certain limit operations, like integration and summation, and in these situations we normally operate under the assumption that such interchanges are permissible.