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## **Supporting conic design methods and conic intersection properties**

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**Abstract.** The supporting ellipsoids and linear programming reflector design methods build upon the property of conics to address the inverse problem of finding the freeform surface that directs light from a point source to produce a prescribed target distribution. We review the properties and main computational limitations of the two methods and show that a fast flux estimation method based on contour detection can be used in combination with the supporting ellipsoid algorithm. Once the intersections between neighboring conic patches on the reflector are known, it is possible to estimate the collected flux using the vertices of the intersection boundary. The advantage of using the intersection method to estimate the flux instead of the more common approach—Monte Carlo ray tracing—is that there is no tradeoff between speed and accuracy. Examples of flux estimation with the intersection method for different target configurations are shown. © 2014 Society of Photo-Optical Instrumentation Engineers (SPIE) [DOI: [10.1117/1.OE.53.3.031306](https://doi.org/10.1117/1.OE.53.3.031306)]

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## 1 Introduction

The problem of deriving the optical surface that redirects light from a point source to a predetermined target has great relevance in the field of illumination design. While real sources are extended sources, the point source problem is of interest, since some real sources can be approximated as point sources, or the point source solution can be used as a starting point from which to derive the extended source solution. If the problem has rotational or translational symmetry, it is straightforward to derive the surface that produces the required illumination,<sup>1</sup> but in general nonrotationally symmetric surfaces are needed. This paper reviews the properties of two reflector design methods that produce a point source solution using conic patches and presents some examples of the application of a flux estimation method that was inspired by conic intersection properties.

## 2 Supporting Ellipsoids Method

The supporting ellipsoid algorithm produces a discrete solution to the point source reflector problem.<sup>2</sup> It can be formulated with ellipsoids for the near-field problem or with paraboloids for the far-field problem. The source and target distributions are sampled with  $N_S$  and  $N_T$  points, respectively, and the reflector is made up of patches of ellipsoids (or paraboloids), each of which directs light from the source to a different target point. We refer to the equation of an ellipse with one focus at the origin as shown in Fig. 1(a) in the form

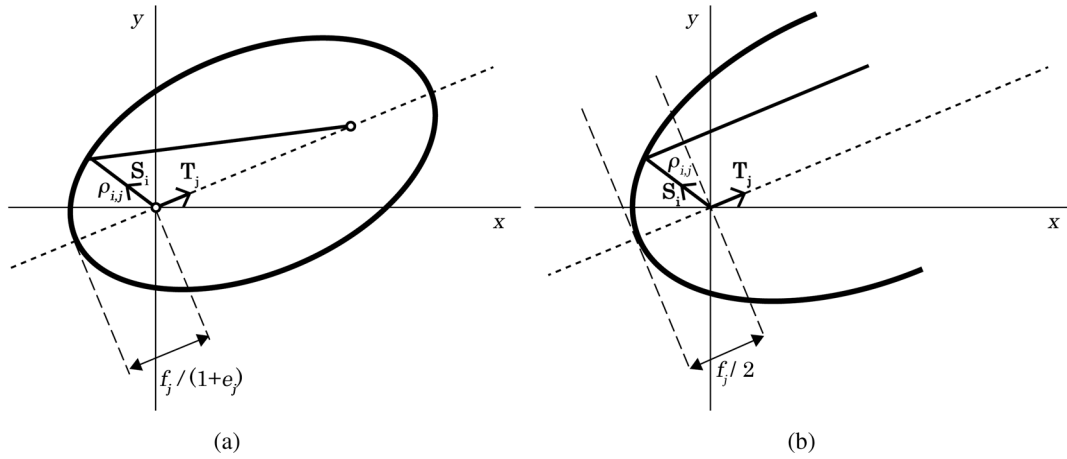
$$\rho_{i,j} = \frac{f_j}{1 - e_j \mathbf{S}_i \cdot \mathbf{T}_j}, \quad (1)$$

with  $i = 1, 2, \dots, N_S$  and  $j = 1, 2, \dots, N_T$ , where  $\rho_{i,j}$  is the distance to the  $j$ 'th ellipse along the  $i$ 'th ray direction,  $f_j$  and  $e_j$  are the focal parameter and eccentricity of the  $j$ 'th ellipse,  $\mathbf{S}_i$  and  $\mathbf{T}_j$  are unit vectors that describe the  $N_S$  source and  $N_T$  target directions, as shown in Fig. 1. The vector  $\mathbf{T}_j$  describes the major axis of the ellipse. A three-dimensional surface is obtained by rotating the ellipse about its major axis, producing an ellipsoid.

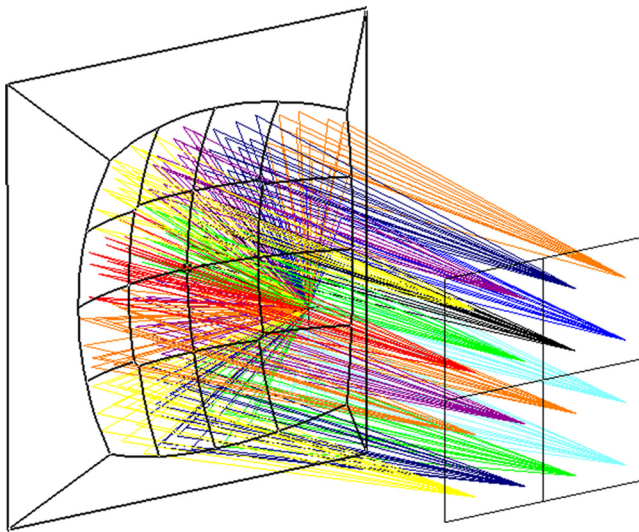
As the eccentricity of the ellipse tends to 1, the ellipsoid becomes a paraboloid with focus at the origin, as shown in Fig. 1(b).

In the following, we refer to the case of ellipsoids, for near-field target illumination; the same method can also be applied to paraboloids for the far-field problem. The iterative supporting ellipsoids method builds upon the imaging property of conics; all ellipsoids are set to have a common focus at the origin, while the other focal point corresponds to a different target point. By varying the parameters of each ellipsoid, it is possible to control the amount of light collected by that ellipsoid and, thus, control how much light is directed to the corresponding target point. An example of a  $4 \times 4$  reflector obtained with the supporting ellipsoids method is shown in Fig. 2; we chose all target points to lie in the same plane and be equally spaced for this example and all the following, but this is not a limitation of the method. Symmetry considerations were not used for this example and the following, but can be leveraged for a faster computation.

In the initial phase of the algorithm, all the flux is collected by one ellipsoid, the reference ellipsoid, whose parameters are chosen to set the scale of the reflector.<sup>3</sup> The remaining ellipsoids are then iteratively scaled until the desired target distribution is produced. The main steps of the supporting ellipsoids algorithm are shown in Fig. 3.



**Fig. 1** (a) Ellipse with one focus at the origin, focal parameter  $f_j$ , eccentricity  $e_j$ , and second focus in direction  $T_j$ . The distance to the parabola along direction  $S_j$  is given by  $\rho_{i,j}$ . (b) Parabola with focus at the origin, focal parameter  $f_j$ , and axial direction  $T_j$ . The distance to the parabola along direction  $S_j$  is given by  $\rho_{i,j}$ .

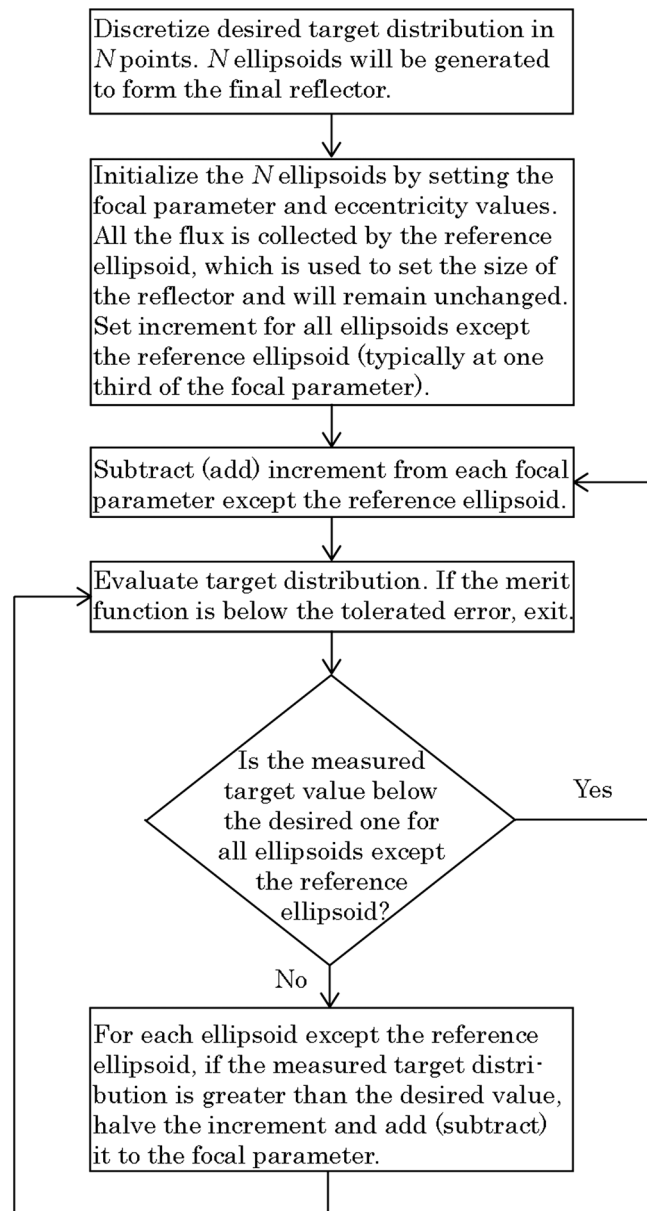


**Fig. 2** Reflector made of  $4 \times 4$  ellipsoid patches obtained with the supporting ellipsoid design method to uniformly illuminate a square target. Each of the 16 ellipsoids reflects rays from the point source to a different target point. The rays collected from different ellipsoids are shown with different colors. The source emission is uniform within a 40-deg square emission in the tangent plane.

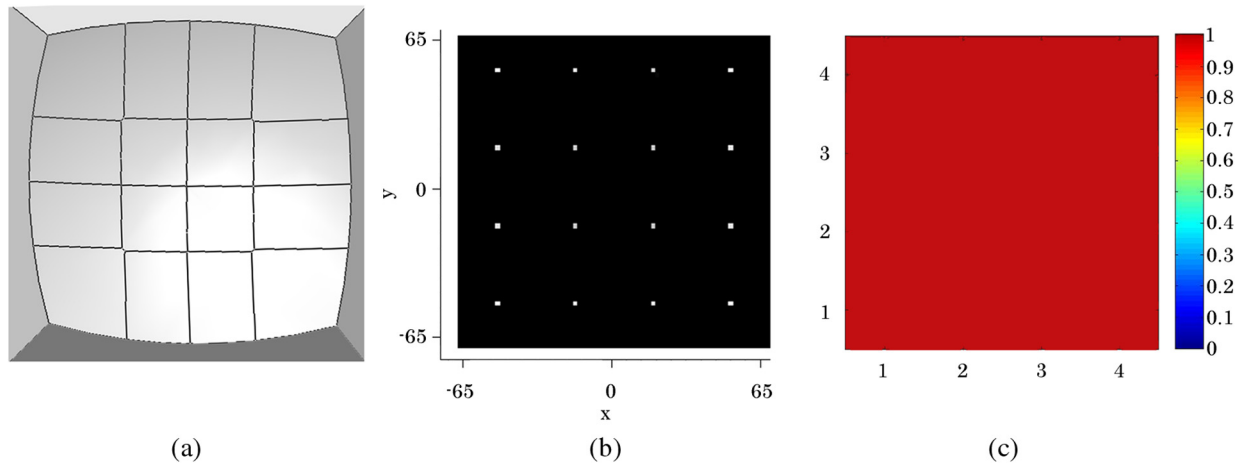
There are two possible geometries for any given problem, converging or diverging. The difference between the two is whether the rays cross each other after reflection by the conic patches. The choice of the desired geometry must be made at the initialization stage of the algorithm.

The exit condition is usually defined by assigning a maximum tolerated error to the merit function. The merit function is typically taken to be the sum of the squared difference between desired and actual normalized target flux.<sup>3</sup>

Commonly, the evaluation of the target distribution is done by Monte Carlo ray tracing. The choice of how many rays to trace affects the speed and the accuracy of the computation; as the number of rays traced per ellipsoid increases, the computational complexity increases linearly, while the accuracy only improves as the square root.<sup>4</sup>



**Fig. 3** Supporting ellipsoids algorithm. The steps are described for the converging (diverging) geometry case.



**Fig. 4** (a) LightTools® model of a  $4 \times 4$  reflector obtained with the supporting ellipsoids method. (b) Target illumination of the discrete spots in the target plane. (c) Raster chart of the normalized flux averaged over each bin corresponding to a discrete target point, evaluated tracing 16,000 rays.

An example of a reflector obtained for a uniform  $4 \times 4$  target derived with the supporting ellipsoids for a uniform source emission over a 40-deg square in the tangent plane is shown in Fig. 4. The discrete target points shown in Fig. 4(b) are each illuminated by a different ellipsoid patch. The raster chart of the flux in Fig. 4(c), obtained by tracing 16,000 rays and averaging the flux collected over each bin, shows that the desired uniform target distribution is achieved.

### 3 Linear Programming Method

A variational formulation of the inverse problem has been proposed independently by Wang and by Olikier.<sup>5,6</sup> In the linear programming method, the reflector problem is formulated as a maximization problem.

$$\max [I_S(S)u(S) + I_T(T)v(T)], \tag{2}$$

with linear constraint

$$u(S) + v(T) \leq -\log(1 - \langle S, T \rangle), \tag{3}$$

where  $I_S(S)$  and  $I_T(T)$  are the normalized source and target intensities in directions  $S$  and  $T$ , respectively,  $\langle \cdot, \cdot \rangle$  denotes

the inner product, and the solutions  $u(S)$  and  $v(T)$  are related to the focal parameters  $f(T)$  and the polar radii  $\rho(S)$  of the paraboloids as

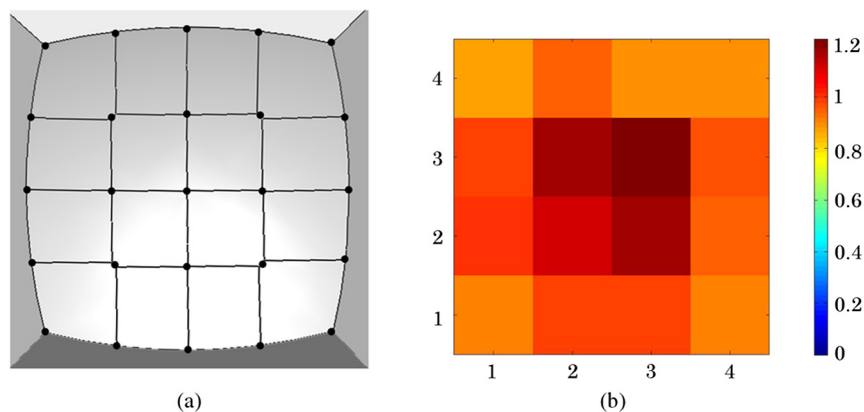
$$u(S) = \log[\rho(S)], \tag{4}$$

$$-v(T) = \log[f(T)]. \tag{5}$$

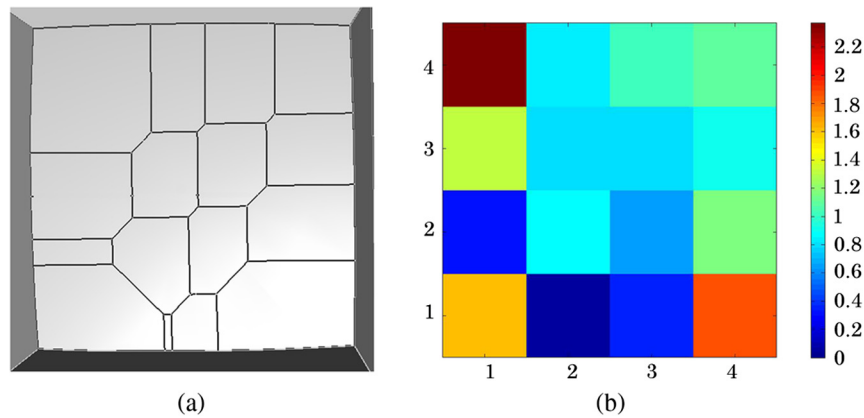
For energy conservation, the total source flux collected by the reflector has to be equal to the total flux at the target.

The linear programming method can be formulated in matrix form, and the resulting matrix has size  $N_S N_T \times (N_S + N_T)$ . As the number of rays and targets increases, the matrix quickly becomes very large, and the computation time increases quadratically.<sup>6,7</sup>

It was shown that there is a special relationship between number of rays and number of targets that makes it possible to obtain the solution with a minimum number of rays equal to  $(N_x + 1)(N_y + 1)$ , where  $N_T = N_x N_y$ .<sup>7</sup> An example of a reflector obtained for a uniform  $4 \times 4$  target and a source emission over a 40-deg square in the tangent plane with the linear programming when the condition is satisfied (i.e., with 25 rays used to sample the source distribution) is shown in Fig. 5.



**Fig. 5** (a) LightTools® model of a  $4 \times 4$  reflector obtained with the linear programming method run with the 25 rays shown as dots on the reflector. (b) Normalized flux evaluated tracing 16,000 rays.



**Fig. 6** (a) LightTools® model of a  $4 \times 4$  reflector obtained with the supporting ellipsoids method run with target evaluation by Monte Carlo ray tracing with 32 rays. (b) Normalized flux evaluated tracing 16,000 rays.

When only  $(N_x + 1)(N_y + 1)$  rays are used, they must be selected according to the mapping between rays and targets, which is in general not known *a priori*.<sup>8</sup> In cases for which the mapping is known, such as the example shown in Fig. 5, the linear programming method offers advantages over the supporting ellipsoids method in terms of speed; an example of the corresponding reflector obtained with the supporting ellipsoids method run with Monte Carlo flux estimation with only two rays/ellipsoids is shown in Fig. 6.

In general, however, given that the mapping between source and targets is not known, the linear programming method is still mainly limited by computational complexity when used to address problems of interest (e.g.,  $N_x, N_y > 10$ ).

#### 4 Intersection of Conics

While the linear programming method has been shown to provide the intersections between conic patches, if the correct ray selection is made,<sup>7</sup> the supporting ellipsoids method provides complementary information—if enough rays are traced, the centroid of each patch can be estimated. However, it was recently shown that by calculating the intersections between neighboring ellipsoids, it is possible to estimate the collected flux, as an alternative to Monte Carlo ray tracing.<sup>9</sup>

If the border of a conic patch is known, it is possible to calculate its area and thus estimate the collected flux.<sup>10</sup> Fournier investigates using ray tracing to identify the border of each facet for flux estimation within the supporting ellipsoids algorithm and shows that this approach has significant overhead compared to Monte Carlo ray tracing.<sup>11</sup>

An alternative way to find the border of each conic patch is to calculate the intersections between neighboring conics. The task of finding the intersection point between quadrics has been investigated in the realm of computer graphics for decades. In general, it is a complex problem for which a robust solution was proposed by Lazard et al.<sup>12,13</sup> The method deals with the general case of finding the intersection curves between two quadric surfaces at an arbitrary location in space. A major drawback is that, because of the generality of the framework needed to address generic quadrics, the computation of the intersection curves is too slow to be used for flux estimation in conjunction with the supporting ellipsoids algorithm. However, given that the geometry of

interest in the methods described in Secs. 2 and 3 is the particular case of ellipsoids or paraboloids sharing a common focus at the origin, for which a simple expression is given in Eq. (1), the intersection between multiple ellipsoids can in fact be calculated analytically.<sup>9</sup>

Under the assumption that the knowledge of neighboring ellipsoids can be determined by the arrangement of the target points, the maximum number of 3-ellipsoid intersections that needs to be calculated is  $4(N_x - 1)(N_y - 1)$ , for a target made of a grid of  $N_x \times N_y$  points.<sup>9</sup> On the other hand, if a brute force Monte Carlo ray tracing approach is used to estimate the flux with  $n$  rays per ellipsoid,  $n(N_x \times N_y)$  ( $N_x \times N_y$ ) ray traces must be performed.

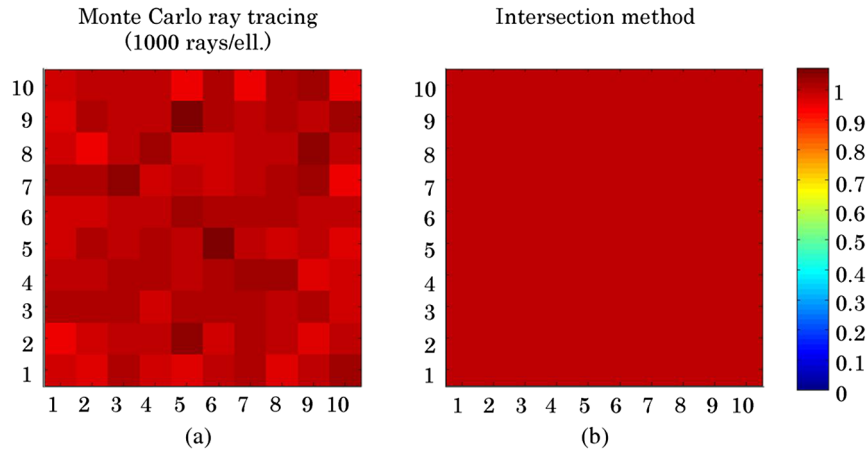
An example of flux estimation for a  $10 \times 10$  reflector obtained with the supporting ellipsoids method with a point source placed at the origin emitting a Lambertian distribution into a 40-deg square in the tangent plane is shown in Fig. 7. The flux estimation done with the intersection method was  $\sim 459$  times faster than Monte Carlo ray tracing with 1000 rays/ellipsoid. In addition to being faster, the intersection method also provides a more accurate result.

The standard deviation of the area estimated with the intersection method in Fig. 7(b) is  $< 0.05\%$ . To achieve a statistical noise of 0.05% with Monte Carlo ray tracing, a number of rays per ellipsoid equal to  $(1/0.0005)^2 = 4$  million would be needed.

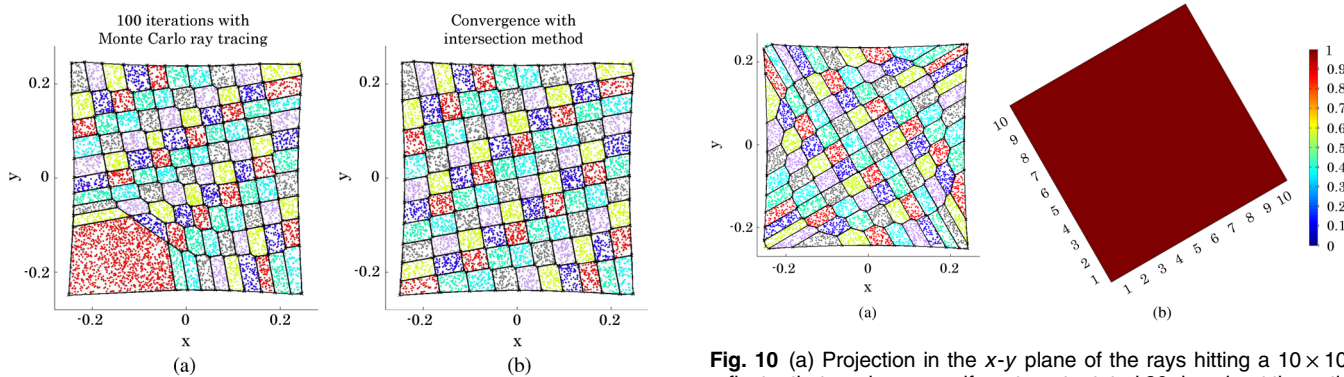
#### 5 Flux Estimation with the Intersection Method

The intersection method can be used in conjunction with the supporting ellipsoids algorithm to provide a faster and more accurate target flux estimation than Monte Carlo ray tracing. Because of the assumptions made about the knowledge of the neighboring ellipsoids from the target layout, the intersection method can be applied once all ellipsoids are supporting and the condition is met. For a uniform  $10 \times 10$  square target grid of width 10 located at  $z = 100$  and rotated 10 deg about the  $z$  axis, we ran the supporting ellipsoids method with flux estimation done by Monte Carlo ray tracing until the intermediate reflector shown in Fig. 8(a) was obtained; we then continued the supporting ellipsoids method switching to the intersection method for flux estimation. The final reflector obtained is shown in Fig. 8(b).

The normalized standard deviation of the target flux calculated with the intersection method was 114% for the



**Fig. 7** Normalized flux calculated (a) with Monte Carlo ray tracing with 1000 rays/ellipsoid and (b) with the intersection method.

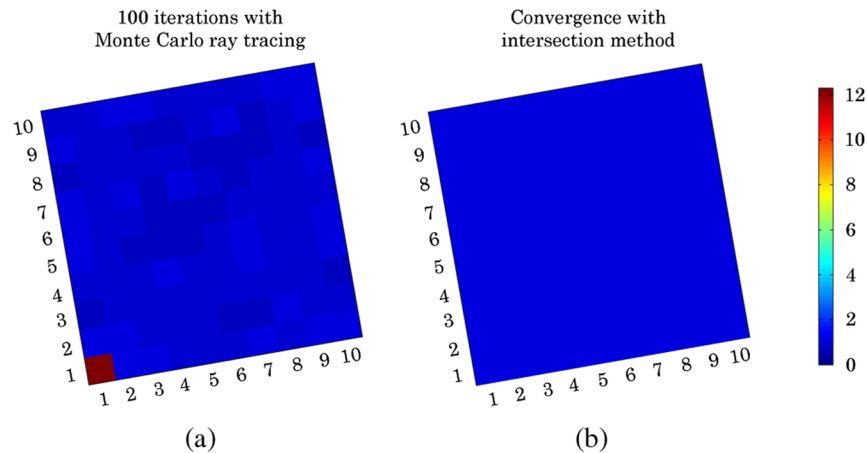


**Fig. 8** Projection in the  $x$ - $y$  plane of the rays hitting the  $10 \times 10$  reflector obtained with the supporting ellipsoids algorithm (a) with Monte Carlo flux estimation for 100 iterations and (b) switching to the intersection method until the final solution is obtained. The target was a uniform grid with a 10-deg twist about the optical axis. Rays hitting different ellipsoids have different colors for visualization purposes; the vertices and boundaries calculated with the intersection method are shown in black.

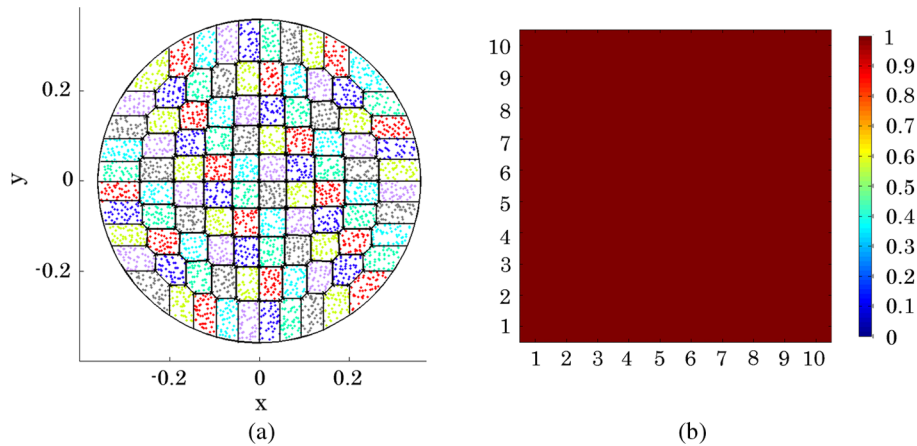
**Fig. 10** (a) Projection in the  $x$ - $y$  plane of the rays hitting a  $10 \times 10$  reflector that produces a uniform target rotated 30 deg about the optical axis. (b) Normalized target flux evaluated with the intersection method.

intermediate reflector in Fig. 8(a) and 0.036% for the final reflector in Fig. 8(b); the target flux estimated with the intersection method for the reflectors of Fig. 8 is shown in Fig. 9.

The intersection method can be applied to estimate the flux of reflectors with conic patches with significantly different shapes, as can be seen in Fig. 10. In this example, the



**Fig. 9** Normalized flux calculated with the intersection method for the reflector of Figs. 8(a) and 8(b), respectively. After convergence, the normalized flux at all target points is one.



**Fig. 11** (a) Projection in the  $x$ - $y$  plane of the rays hitting a  $10 \times 10$  reflector that produces a uniform target from a circular source distribution. (b) Normalized target flux evaluated with the intersection method.

reflector produced a  $10 \times 10$  discrete uniform target that is rotated 30 deg about the optical axis. A raster chart of the flux is shown in Fig. 10(b); the target flux was nearly perfectly uniform, with normalized standard deviation well below  $10^{-5}$ .

An example of a circle-to-square reflector and the normalized flux of the target are shown in Fig. 11. The source had a Lambertian distribution over a 40-deg cone in cosine space. The target flux produced by the reflector was nearly perfectly uniform, with normalized standard deviation well below  $10^{-5}$ .

## 6 Conclusion

The linear programming and supporting ellipsoids methods for reflector design exploit the properties of conics to build a reflector surface that solves the point source problem. The intersection method, a fast and accurate flux estimation method for conic patches of a reflector based on the calculation of the intersections between neighboring ellipsoids, can be used in conjunction with the supporting ellipsoids design algorithm for faster flux estimation than what is possible with the more commonly used Monte Carlo ray tracing. Thanks to its linear dependence on the number of targets, this flux estimation method is more scalable than Monte Carlo ray tracing, which has a quadratic dependence. Additionally, the intersection method provides an accurate result without having to compromise computational speed.

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