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Abstract. A first-order formula for calculations of the direction cosines of the rays refracted by Risley prisms was derived. The formula was obtained by representing the deviation of the ray passing through a prism by the product of rotation matrices, and using the series expansion of the product. It can be applied to the system of an arbitrary number of prisms or a combination of Risley prisms. Related errors were discussed and some numerical calculations were made and compared with the exact solutions using the refraction equation. The scan patterns of a single Risley prism or a combination of two Risley prisms calculated using the generalized first-order formula are in good agreement with the exact solutions. © 2011 Society of Photo-Optical Instrumentation Engineers (SPIE). [DOI: 10.1117/1.3655501]

Subject terms: Risley prism; deviation; beam scanning; scan pattern; first-order approximation.

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1 Introduction

Risley prisms, composed of two prisms with a small apex angle, are widely used for beam scanning or steering in optical instruments[1–3] and other developing systems.[4–7] The beam steering using three prisms is also investigated for the same applications.[7] The primary concern for analyses of Risley prisms is to calculate the deviations of the ray passing through them, thereby obtaining the steering or scan patterns. The basic properties of Risley prisms can be understood as the combination of deviations by each prism,[8,9] but the rigorous calculation is complicated because several refractions at planar surfaces are involved. In previous works, several methods have been used such as the three-dimensional model,[2] analytic formula,[10] and approximate formula.[11,12] Recently, approximate formulas up to third-order of the apex angle were obtained by expanding analytic solution.[13] But all the analytic formulas and the approximate ones in those works were obtained for a single Risley prism composed of two prisms. In this paper, by representing the deviation of a ray passing through a prism by the product of rotation matrices, a generalized first-order formula was obtained. It can be applied to the system of an arbitrary number of prisms or combination of Risley prisms. Related errors were discussed and some numerical calculations were made and compared with the exact solutions using the refraction equation. The scan patterns of a single Risley prism or a combination of two Risley prisms were calculated using the generalized first-order formula, and the results were in good agreement with the exact solutions.

2 Refraction Equation for Cascaded Planar Surfaces

Let $\mathbf{s}_i$ and $\mathbf{s}_i'$ be unit vectors in the direction of incident and transmitted rays at i'th surface, and $\mathbf{N}_i = (\sin \alpha_i \cos \phi_i, \sin \alpha_i \sin \phi_i, \cos \alpha_i)$ is the unit normal vector at this surface (see Fig. 1), then Snell's law can be written as[14]

$$n_i \mathbf{N}_i \times \mathbf{s}_i = n_i' \mathbf{N}_i \times \mathbf{s}_i', \tag{1}$$

where n_i and n_i' are refractive indices of the mediums before and after the i'th surface. Performing vector product with $\mathbf{N}_i$ on both sides of Eq. (1) and using the vector identity: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$, we obtain

$$n_i[(\mathbf{N}_i \cdot \mathbf{s}_i)\mathbf{N}_i - \mathbf{s}_i] = n_i'[(\mathbf{N}_i \cdot \mathbf{s}_i')\mathbf{N}_i - \mathbf{s}_i']. \tag{2}$$

This gives

$$\mathbf{s}_i' = \gamma_i' \mathbf{N}_i - \frac{n_i}{n_i'}(\gamma_i \mathbf{N}_i - \mathbf{s}_i), \tag{3}$$

where the following definitions are used:

$$\gamma_i \equiv \mathbf{N}_i \cdot \mathbf{s}_i = \cos \Theta_i, \quad \gamma_i' \equiv \mathbf{N}_i \cdot \mathbf{s}_i' = \cos \Theta_i', \tag{4}$$

where Θ_i is the angle between $\mathbf{N}_i$ and $\mathbf{s}_i$ and Θ_i' is defined similarly. From Snell's law: $n_i' \sin \Theta_i' = n_i \sin \Theta_i$ and Eq. (4), we have

$$\gamma_i' = \sqrt{1 - \sin^2 \Theta_i'} = \frac{1}{n_i'}\sqrt{n_i'^2 - n_i^2 + n_i^2 \gamma_i^2}. \tag{5}$$

Equation (3), with coefficients given by Eq. (5), is the refraction equation at i'th surface. For successive calculations, we put $\mathbf{s}_{i+1} = \mathbf{s}_i'$ and $n_{i+1} = n_i'$. By applying the refraction equation successively to each surface, we can obtain directions of a ray passing through an arbitrary number of surfaces. We notice that the components of the ray vector $\mathbf{s}_i = (s_{xi}, s_{yi}, s_{zi})$ are direction cosines of the ray. The components of the refracted ray vector are represented as $\mathbf{s}_i' = (s_{xi}', s_{yi}', s_{zi}')$. Normal vectors and ray vectors for analysis of a typical Risley prism are shown in Fig. 2. We define the rotation angle φ_i of each prism to be zero when the apex is directed downward along the x axis as shown in Fig. 2. When the second and third surfaces are orthogonal to the z axis, the azimuth angles of normal vectors $\mathbf{N}_1$ and $\mathbf{N}_4$ are related to the rotation angles by

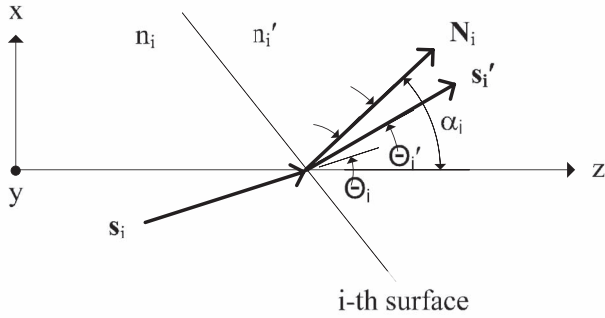$$\phi_1 = \varphi_1, \quad \phi_4 = \pi + \varphi_2. \tag{6}$$

Fig. 1 Notations related to the refraction equation.

For the configuration of Fig. 2 with $\varphi_2 = \pi$, we have $\phi_4 = 2\pi$, which means $\mathbf{N}_4 = \mathbf{N}_1$ when the apex angles are same, as expected. In Sec. 4, we will use Eqs. (3)–(5) for numerical calculations.

Substituting Eq. (5) into Eq. (3) gives $\mathbf{s}'_i = n\mathbf{s}_i + \sqrt{1 - n^2 + n^2\gamma_i^2}\mathbf{N}_i - n\gamma_i\mathbf{N}_i$ where n is defined by $n = n_i/n'_i$. The refraction equation in this form is used for analysis of the Risley prisms,[13] and also for analyses of the plano convex or hyperboloidal focusing lenses.[15,16]

3 Derivation of the Approximate Formula

By applying Eqs. (3)–(5) to two successive surfaces of a prism with refractive index n, we obtain

$$\mathbf{s}'_2 = (\gamma'_2 - n\gamma_2)\mathbf{N}_2 + (n\gamma'_1 - \gamma_1)\mathbf{N}_1 + \mathbf{s}_1, \tag{7}$$

which means the vector $\mathbf{s}'_2 - \mathbf{s}_1$ is in the plane generated by $\mathbf{N}_1$ and $\mathbf{N}_2$. So, if $\mathbf{N}_1$ and $\mathbf{N}_2$ are in the x-z plane, then the effect of the prism is the rotation of the ray vector $\mathbf{s}_1$ about the y-axis, and it is clear that the rotation is counter-clockwise when the rotation angle φ is zero. Let δ be the deviation angle of the prism, then this rotation can be represented in matrix form as follows[17]

$$R_y(\delta) = \begin{pmatrix} \cos\delta & 0 & \sin\delta \\ 0 & 1 & 0 \\ -\sin\delta & 0 & \cos\delta \end{pmatrix}. \tag{8}$$
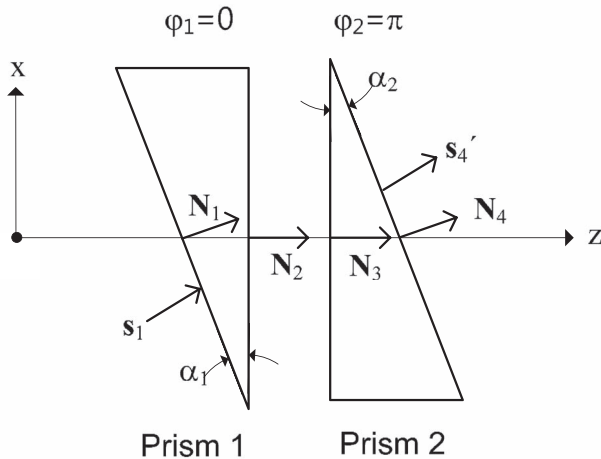

Fig. 2 Normal vectors and ray vectors for analysis of a Risley prism; the rotation angles are $\varphi_1 = 0$ and $\varphi_2 = \pi$.

To find the transmitted ray vector for a prism rotated by φ in azimuth, the components of the incident ray vector have to be transformed to the coordinate system with $\varphi = 0$, i.e., the coordinate system fixed to the prism, then rotated about the y axis by δ and transformed to the original coordinate system. This operation can be represented as follows:

$$M_\delta(\varphi) = R_z(\varphi)R_y(\delta)R_z(-\varphi). \tag{9}$$

Here the matrix $R_z(\varphi)$ for the rotation about the z axis is

$$R_z(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

The incident ray vector $\mathbf{s}_1 = (s_{1x}, s_{1y}, s_{1z})$ and the transmitted ray vector $\mathbf{s} = (s_x, s_y, s_z)$ which is equal to $\mathbf{s}'_2$ of Eq. (8) are related by

$$\mathbf{s}^T = M_\delta(\varphi)\mathbf{s}_1^T, \tag{11}$$

where the superscript T represents the transpose operation for the components of vector. Substituting Eq. (8) into Eq. (9), and using the series expansion about δ gives

$$M_\delta(\varphi) = M_\delta^{(1)}(\varphi) - \frac{1}{2}\delta^2 M_\delta^{(2)}(\varphi) + O(\delta^3)I, \tag{12}$$

where the matrices are defined by

$$M_\delta^{(1)}(\varphi) = \begin{pmatrix} 1 & 0 & \delta\cos\varphi \\ 0 & 1 & \delta\sin\varphi \\ -\delta\cos\varphi & -\delta\sin\varphi & 1 \end{pmatrix}, \tag{13}$$

and

$$M_\delta^{(2)}(\varphi) = \begin{pmatrix} \cos^2\varphi & \cos\varphi\sin\varphi & 0 \\ \cos\varphi\sin\varphi & \sin^2\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{14}$$

and I in Eq. (12) is the unit matrix.

When the incident ray is in the x-z plane, i.e., $s_{1y} = 0$, the deviation angle δ is given by[18]

$$\delta = \Theta - \alpha + \sin^{-1}[\sin\alpha \cdot (n^2 - \sin^2\Theta)^{1/2} - \sin\Theta\cos\alpha]. \tag{15}$$

When s_{1y} is not zero, the deviation angle must be defined to be the angle between the vectors projected on the x-z plane, i.e., $\mathbf{s}_{1p} \equiv (s_{1x}, 0, s_{1z})$ and $\mathbf{s}_p \equiv (s_x, 0, s_z)$. If the angle between $\mathbf{s}_{1p}$ and $\mathbf{N}_1$ is denoted by Θ_p, and the angle between $\mathbf{s}_{1p}$ and $\mathbf{s}_1$ by Θ_v as shown in Fig. 3, then the corresponding angles Θ'_p and Θ'_v for the refracted ray $\mathbf{s}'_1$ are given by Snell's law in the following form:[19]

$$n_r \sin\Theta_p \cos\Theta_v = n'_r \sin\Theta'_p \cos\Theta'_v \tag{16a}$$

$$n_r \sin\Theta_v = n'_r \sin\Theta'_v, \tag{16b}$$

where n_r and n'_r are refractive indices of the mediums before and after the surface. By using Eq. (16a) with $n_r = 1$ and $n'_r = n$, and following the same method for derivation of Eq. (15), we obtain

$$\delta = \Theta_p - \alpha + \sin^{-1}[\sin\alpha \cdot (n^2\psi^2$$
$$- \sin^2\Theta_p)^{1/2} - \sin\Theta_p\cos\alpha], \tag{17}$$

where $\psi = \cos\Theta'_v / \cos\Theta_v$. Using Eq. (16b), we have $\psi \approx 1 + [(n^2 - 1)/(2n^2)]\Theta_v^2$. When Θ_v is zero, we have Θ_p

$= \Theta$ and Eq. (17) reduces to Eq. (15). By expanding Eq. (17) up to the third-order terms, we obtain

$$\delta = (n-1)\alpha + (\nu/n)\alpha\Theta_p^2 - \nu\alpha^2\Theta_p$$
$$+ (\nu/n)\alpha\Theta_v^2 + (n/3)\nu\alpha^3, \tag{18}$$

where the constant ν is defined by $\nu = (1/2)(n^2 - 1)$. The first-order approximation of δ is $\delta^{(1)} = (n-1)\alpha$ as known, and Eq. (18) can be written as $\delta = \delta^{(1)} + \epsilon^{(3)}$, where $\epsilon^{(3)}$ is the third-order terms:

$$\varepsilon^{(3)} = (\nu/n)\alpha\Theta_p^2 - \nu\alpha^2\Theta_p + (\nu/n)\alpha\Theta_v^2 + (n/3)\nu\alpha^3. \tag{19}$$

$$s_x = s_{1x} + \delta\cos\varphi \cdot s_{1z} - (1/2)\delta^2(s_{1x}\cos^2\beta + s_{1y}\cos\beta\sin\beta) + O(\delta^3)s_{1x},$$
$$s_y = s_{1y} + \delta\sin\varphi \cdot s_{1z} - (1/2)\delta^2(s_{1x}\cos\beta\sin\beta + s_{1y}\sin^2\beta) + O(\delta^3)s_{1y}, \tag{20}$$
$$s_z = s_{1z} - \delta\cos\varphi \cdot s_{1x} - \delta\sin\varphi \cdot s_{1y} - (1/2)\delta^2 s_{1z} + O(\delta^3)s_{1z}.$$
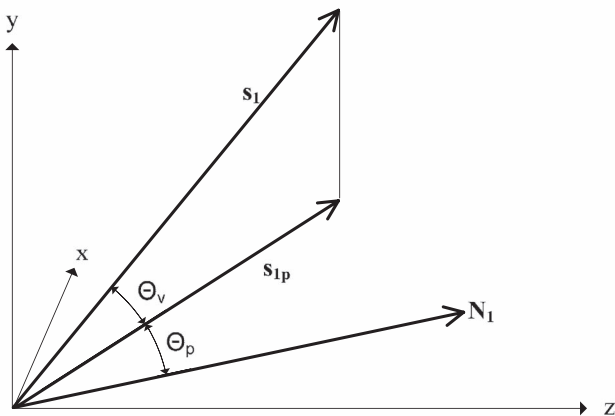
Using $\delta = \delta^{(1)} + \epsilon^{(3)}$, and $s_{1x} \approx s_{1y} \approx \Theta$ and $s_{1z} \approx 1$ in the higher order terms, Eq. (20) becomes

$$s_x = s_{1x} + \delta^{(1)}\cos\varphi \cdot s_{1z} + \varepsilon^{(3)}\cos\varphi \cdot s_{1z} + O((n-1)^2\alpha^2\Theta),$$
$$s_y = s_{1y} + \delta^{(1)}\sin\varphi \cdot s_{1z} + \varepsilon^{(3)}\cos\varphi \cdot s_{1z} + O((n-1)^2\alpha^2\Theta), \tag{21}$$
$$s_z = s_{1z} - \delta^{(1)}\cos\varphi \cdot s_{1x} - \delta^{(1)}\sin\varphi \cdot s_{1y} + O((n-1)^2\alpha^2).$$

Equation (21) means that if we use the first-order approximation $M_\delta(\varphi)$, i.e.,

$$M_\delta(\varphi) = \begin{pmatrix} 1 & 0 & \delta^{(1)}\cos\varphi \\ 0 & 1 & \delta^{(1)}\sin\varphi \\ -\delta^{(1)}\cos\varphi & -\delta^{(1)}\sin\varphi & 1 \end{pmatrix}, \tag{22}$$

then the errors in calculating s_x and s_y are determined by the third-order terms of α and Θ, and the one for s_z is determined

Fig. 3 Geometry for definitions of incident angles Θ_p and Θ_v for an oblique ray vector $\mathbf{s}_1$. Here the vectors $\mathbf{s}_{1p}$ and $\mathbf{N}_1$ are on the x-z plane.

Figure 4 shows the deviation angles calculated by Eqs. (17) and (18). It is for the case of the apex angle $\alpha = 0.2$ rad (≈ 11.5 deg) and the refractive index $n = 1.5$, so that $\delta^{(1)} = 0.1$ rad. It is seen that the errors of the first-order approximation are in the range of 0.6 to 2.5 mrad at $\Theta_v = 0.0$ rad, and 1.5 to 3.5 mrad at $\Theta_v = 0.1$ rad. The graph of the third-order approximation is symmetric about $\Theta_p = 0.15$ rad because $\epsilon^{(3)}$ is the quadratic equation of Θ_p and it has the minimum value at $\Theta_p = (1/2)n\alpha = 0.15$ rad which is the approximate value of the minimum deviation angle.

By substituting Eqs. (13) and (14) into Eq. (12), the components of $\mathbf{s}$ can be written as

by the second-order term of α. The last term in the equation of $\mathbf{s}_x$ or $\mathbf{s}_y$ in Eq. (21) depends on Θ, so that the errors do not vanish even in the case of $\epsilon^{(3)} = 0$. For example, when $n = 1.5$, $\alpha = 0.2$ rad, and $\Theta = 0.1$, it is $(n-1)^2\alpha^2\Theta = 1.0$ mrad. The order of magnitude of the third terms including $\varepsilon^{(3)}$ are estimated by assuming $\varphi = 0$ rad and $s_{1z} = 1$. Figure 4 shows that $\varepsilon^{(3)} \approx 1.7$ mrad when $\Theta_p = \Theta_v = 0.1$ rad, from which the total error in calculation of $\mathbf{s}_x$ or $\mathbf{s}_y$ using the first-order formula is estimated to be 2.7 mrad. When several prisms are involved, the total error depends on the relative orientations of the prisms, and the error analysis done here will give only the order of magnitudes.

Hereafter, we will use Eq. (22) for calculations of ray vectors, and also use $\delta = \delta^{(1)}$ by dropping the upper index. To investigate the scan patterns, only s_x and s_y are needed, so that the errors of our first-order formula are of third-order.

For analysis of a Risley prism, let the deviation angle of the first prism be δ_1, and the one of the second prism be δ_2, and the rotation angle of each prism be φ_1 and φ_2. Since $M_\delta(\varphi)$ in Eq. (22) is independent of incident ray vectors, the transmitted ray vector $\mathbf{s}$ can be obtained from the following equation:

$$\mathbf{s}^T = M_{\delta_2}(\varphi_2)M_{\delta_1}(\varphi_1)\mathbf{s}_1^T \equiv \bar{M}\mathbf{s}_1^T. \tag{23}$$

Substituting Eq. (22) into (23) and keeping the terms of the first-order with respect to δ_1 or δ_2, we have

$$\bar{M} = \begin{pmatrix} 1 & 0 & \delta_1 \cos\varphi_1 + \delta_2 \cos\varphi_2 \\ 0 & 1 & \delta_1 \sin\varphi_1 + \delta_2 \sin\varphi_2 \\ -\delta_1 \cos\varphi_1 - \delta_2 \cos\varphi_2 & -\delta_1 \sin\varphi_1 - \delta_2 \sin\varphi_2 & 1 \end{pmatrix}. \tag{24}$$

Using Eq. (23) gives

$$
\begin{aligned}
s_x &= s_{x1} + (\delta_1 \cos\varphi_1 + \delta_2 \cos\varphi_2)s_{z1}, \\
s_y &= s_{y1} + (\delta_1 \sin\varphi_1 + \delta_2 \sin\varphi_2)s_{z1}, \\
s_z &= -(\delta_1 \cos\varphi_1 + \delta_2 \cos\varphi_2)s_{x1} \\
&\quad -(\delta_1 \sin\varphi_1 + \delta_2 \sin\varphi_2)s_{y1} + s_{z1}.
\end{aligned}
\tag{25}
$$

Equation (25) is the first-order formula for the beam deviation of an arbitrary incident ray $\mathbf{s}_1$. There is no restriction on the incident ray $\mathbf{s}_1$ in Eq. (25), so that it can be applied to the axial ray or oblique rays. When the incident ray is axial; $\mathbf{s}_1 = (0, 0, 1)$, Eq. (25) gives

$$
\begin{aligned}
s_x &= \delta_1 \cos\varphi_1 + \delta_2 \cos\varphi_2, \\
s_y &= \delta_1 \sin\varphi_1 + \delta_2 \sin\varphi_2, \\
s_z &= 1 .
\end{aligned}
\tag{26}
$$

Equation (26) has been used for analyses of scan patterns of Risley prisms.[8,9,12]

When $\delta_2 = \delta_1$, $\varphi_1 = 0$, and $\varphi_2 = \varphi'$, Eq. (26) gives

$$s_x = \delta_1(1 + \cos\varphi'), \quad s_y = \delta_1 \sin\varphi', \quad s_z = 1. \tag{27}$$

A formula equivalent to Eq. (27) can be obtained by using the polar angle θ and the azimuth angle χ of the ray vector $\mathbf{s} = (s_x, s_y, s_z)$. Since $\sin^2\theta = s_x^2 + s_y^2$ and $\sin^2\theta \approx \theta^2$, Eq. (27) gives

$$\theta \approx \sqrt{2\delta_1^2(1 + \cos\varphi')} = 2\delta_1 \cos\frac{\varphi'}{2}, \tag{28}$$

and since $\tan\chi = s_y / s_x$, it gives

$$\tan\chi = \frac{\sin\varphi'}{1 + \cos\varphi'} = \tan\frac{\varphi'}{2}. \tag{29}$$

Equations (28) and (29) were obtained directly by the vector-summation of the deviations.[8]
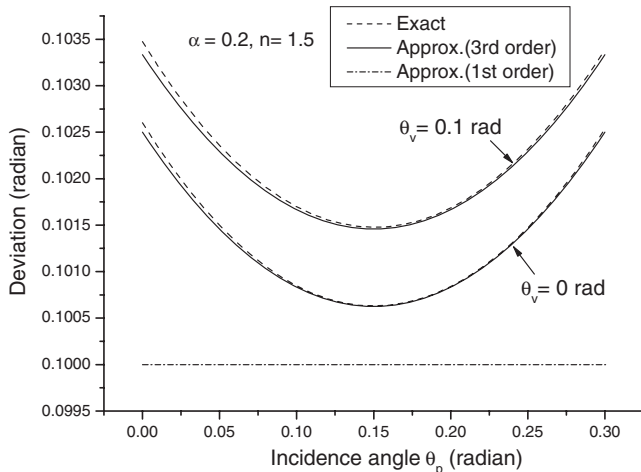

Fig. 4 Deviation angles of a single prism: $\alpha = 0.2$ rad, $n = 1.5$.

It is straightforward to generalize Eq. (25) to the formula for a system composed of arbitrary number of prisms, and the result is

$$
\begin{aligned}
s_x &= s_{x1} + \left(\sum_{i=1}^{N} \delta_i \cos\varphi_i\right) s_{z1}, \\
s_y &= s_{y1} + \left(\sum_{i=1}^{N} \delta_i \sin\varphi_i\right) s_{z1}, \\
s_z &= -\left(\sum_{i=1}^{N} \delta_i \cos\varphi_i\right) s_{x1} \\
&\quad -\left(\sum_{i=1}^{N} \delta_i \sin\varphi_i\right) s_{y1} + s_{z1} ,
\end{aligned}
\tag{30}
$$

where N is the number of prisms, and δ_i is the deviation angle, and φ_i is the rotation angle for the i'th prism.

4 Numerical Calculations for Sample Cases

To obtain the scan patterns of a Risley prism, we put the rotational frequencies of the prisms to be f_1 and f_2 each so that the angles of rotations are

$$\varphi_1(t) = \varphi_1^{(i)} + 2\pi f_1 t, \quad \varphi_2(t) = \varphi_2^{(i)} + 2\pi f_2 t, \tag{31}$$

with time t and initial angles $\varphi_1^{(i)}$ and $\varphi_2^{(i)}$. The scan patterns depend on the initial angles of rotations and the following ratios:[11]

$$k \equiv \delta_2/\delta_1, \quad M \equiv f_2/f_1. \tag{32}$$

For the first case, we consider the configuration where the second prism rotates in the opposite direction with the same frequency, i.e., $f_2 = -f_1$, and the initial angles are $\varphi_1^{(i)} = 0$ and $\varphi_2^{(i)} = \pi$ so that we have $\varphi_2(t) = \pi - \varphi_1(t)$. Let the deviation angles be the same; $\delta_2 = \delta_1$. This case corresponds to $k = 1$ and $M = -1$. From Eq. (26), we obtain the approximate solutions for the components of the transmitted ray vector as follows:

$$s_x = 0, \quad s_y = 2\delta_1 \sin(2\pi f_1 t), \quad s_z = 1, \tag{33}$$

which means the linear scan along the y axis. In this paper, all the refractive indices are assumed to be $n = 1.5$, and the apex angle of prism 1 to be $\alpha_1 = 0.2$ rad, so that $\delta_1 = (n - 1)\alpha_1 = 0.1$ rad. The exact numerical calculation using Eqs. (3)–(5) can be performed straightforwardly, where the azimuth angles of normal vectors are determined by Eq. (6). Figure 5 shows the scan patterns generated during one period of rotation for prism 1. The maximum error in the x direction is about 1.6 mrad at $s_y \approx 0.17$, which is comparable with the total error (=2.7 mrad) obtained for a single prism in
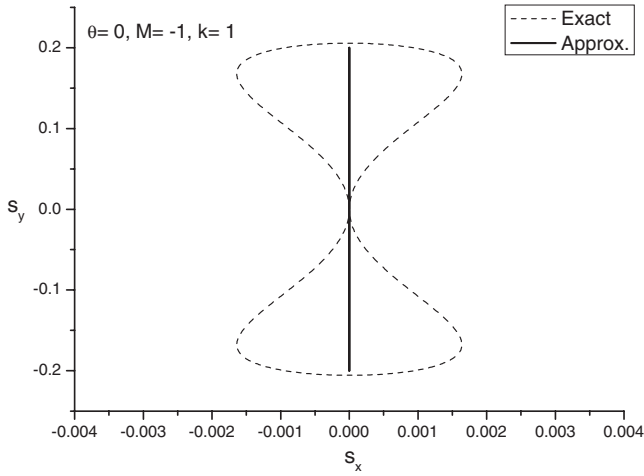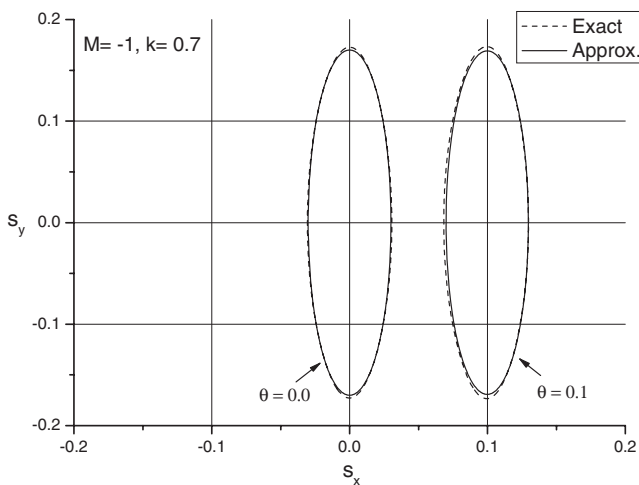
Fig. 5 Scan patterns of Risley prism : $\varphi_1^{(i)} = 0$, $\varphi_2^{(i)} = \pi$, $M = -1$, $k = 1$.

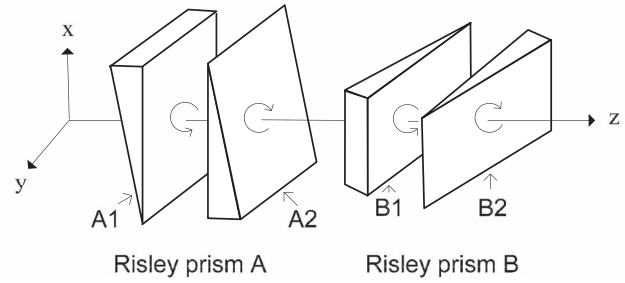**Fig. 7** Geometry of a combination of two Risley prisms.

Sec. 3. The bow tie pattern of the exact solution is one of the typical properties of the Risley prisms with the same apex angle.[20]

Figure 6 is an example of scan patterns obtained by the approximate calculations using Eq. (25) and the exact numerical calculations using Eqs. (3)–(5), in which we consider the axial and the oblique incident rays which are specified by $\mathbf{s}_1 = (\sin\theta, 0, \cos\theta)$ with the polar angles $\theta = 0.0$ and 0.1 rad each. The case of $\theta = 0.0$ rad is comparable to the one given in Ref. 11. It can be seen that the error of the approximate solution for $\theta = 0.0$ rad at $s_x = 0.0$ and $s_y = 0.17$ is approximately 2.8 mrad in the y direction, and the one for $\theta = 0.1$ rad is approximately 4.2 mrad in the x direction. The errors in the x direction are approximately zero in both cases. The approximate solutions are in reasonably good agreement with the exact solutions even though the errors increase with the polar angle θ.

When two Risley prisms are combined as in Fig. 7, it can generate more general two-dimensional scan patterns. As an example, we use the configuration in which the second Risley prism is rotated by 90 deg, and the apex angles of prisms in

each Risley prism are the same. Here the directions of rotation in each Risley prism are opposite. Let the deviation angles of the prisms in each Risley prism be δ_A and δ_B, and the rotation angle be φ_A and φ_B. Therefore we put $\delta_1 = \delta_2 = \delta_A$, $\varphi_1 = \varphi_A$, $\varphi_2 = \pi - \varphi_A$ for Risley prism A. Using Eq. (24), we obtain

$$\bar{M}_A(\varphi_A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2\delta_A \sin \varphi_A \\ 0 & -2\delta_A \sin \varphi_A & 1 \end{pmatrix}. \tag{34}$$

For Risley prism B, we put $\delta_1 = \delta_2 = \delta_B$, $\varphi_1 = (\pi/2) + \varphi_B$, $\varphi_2 = -(\pi/2) - \varphi_B$, so that we obtain

$$\bar{M}_B(\varphi_B) = \begin{pmatrix} 1 & 0 & -2\delta_B \cos \varphi_B \\ 0 & 1 & 0 \\ 2\delta_B \cos \varphi_B & 0 & 1 \end{pmatrix}. \tag{35}$$

The transmitted ray vector $\mathbf{s} = (s_x, s_y, s_z)$, when the incident ray vector is $\mathbf{s}_1$, is given by the following equation:

$$\mathbf{s}^T = M_{\delta_B}(\varphi_B) M_{\delta_A}(\varphi_A) \mathbf{s}_1^T. \tag{36}$$

When the incident ray is axial, substituting Eqs. (34) and (35) into Eq. (36) gives

$$s_x = -2\delta_B \sin \varphi_B, \quad s_y = 2\delta_A \sin \varphi_A, \quad s_z = 1. \tag{37}$$

To obtain a scan pattern, we put the rotational frequencies to be f_A and f_B for each Risley prism so that

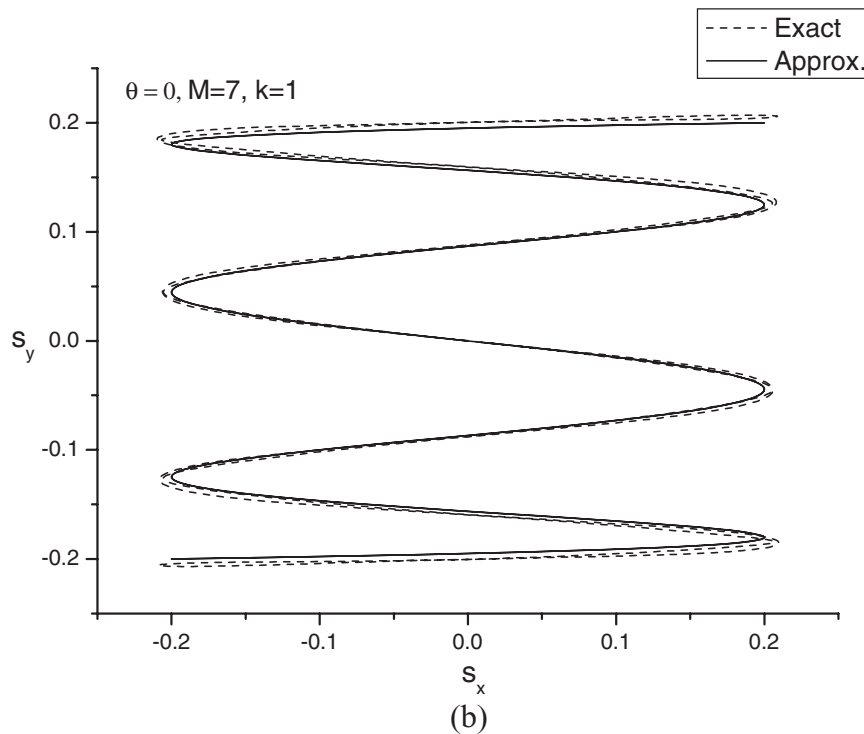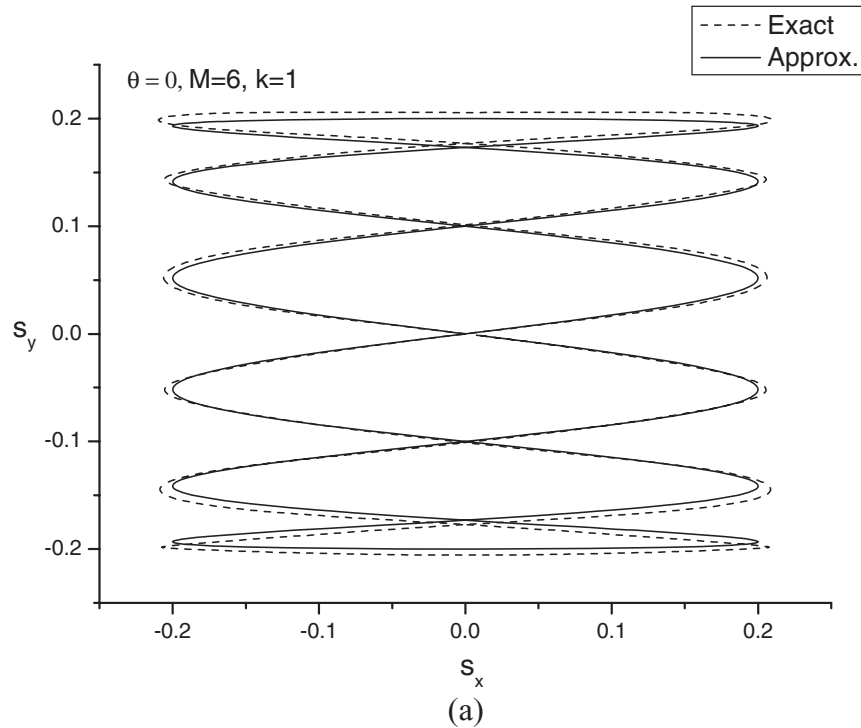$$\varphi_A(t) = 2\pi f_A t, \quad \varphi_B(t) = 2\pi f_B t. \tag{38}$$

The ratios similar to the ones in Eq. (32) can be defined by

$$k \equiv \delta_B/\delta_A, \quad M \equiv f_B/f_A. \tag{39}$$

For exact calculation using the refraction equation, we need to specify the rotation angles of each prism. Let φ_{A1} and φ_{A2} be rotation angles of prisms in Risley prism A (referring to Fig. 7), and φ_{B1} and φ_{B2} be the ones of prisms in Risley prism B, then we have

$$\varphi_{A1}(t) = 2\pi f_A t, \quad \varphi_{A2}(t) = \pi - 2\pi f_A t,$$

$$\varphi_{B1}(t) = \frac{\pi}{2} + 2\pi f_B t, \tag{40}$$

$$\varphi_{B2}(t) = -\frac{\pi}{2} - 2\pi f_B t.$$

Using Eqs. (6) and (40), we can determine all the components of the normal vectors, and perform exact calculations using Eqs. (3)–(5).

Fig. 6 Scan patterns of Risley prism for axial ($\theta = 0$) and oblique ($\theta = 0.1$ rad) incident rays: $\varphi_1^{(i)} = 0$, $\varphi_2^{(i)} = \pi$, $M = -1$, $k = 0.7$.

(a)

(b)

Fig. 8 Scan patterns of a combination of Risley prisms: $\varphi_{A1}^{(i)} = 0$, $\varphi_{A2}^{(i)} = \pi$, $\varphi_{B1}^{(i)} = \pi/2$, $\varphi_{B2}^{(i)} = -\pi/2$. (a) $M = 6$, $k = 1$, (b) $M = 7$, $k = 1$.

Figure 8 shows the scan patterns generated during one period of rotation of the prism A1 for the two cases of $M = 6$, $k = 1$, and $M = 7$, $k = 1$.

It can be seen that the errors of the approximate solutions for $M = 6$ are about 10 mrad in the x direction and 7 mrad in the y direction at the point of $\mathbf{s}_x = 0.2$ and $\mathbf{s}_y = 0.2$. The errors of the same level of magnitudes are obtained for the case of $M = 7$. We can notice that the approximate solutions

have reasonable accuracies for describing the scan patterns. It is also observed that the scan patterns obtained from this configuration are composed of closed curves when M is even.

5 Conclusion

A first-order formula for calculations of the direction cosines of the rays refracted by Risley prisms was derived. The

formula was obtained by representing the deviation of the ray passing through a prism by the product of rotation matrices, and using the series expansion of the product. It can be applied to the system of arbitrary number of prisms or combination of Risley prisms. It permits the calculations of the direction cosines of the transmitted ray vectors for arbitrary incident rays such as oblique rays. The errors associated with the first-order formula were analyzed by using the series expansion of the expression for the deviation angle. It showed that the errors are of third-order of the prism's apex angle and the incidence angle. The numerical estimation using examples showed that the total error for a single prism is reasonably small, approximately 2.7 mrad for incident angles of 0.2 rad. The generalized first-order formula was applied to the numerical calculations of the scan patterns of a single Risley prism and a combination of two Risley prisms, and the results were compared with the exact solutions using the formulation based on the refraction equations. The maximum error in the scan patterns of the single Risley prism in the example was 2.8 mrad for axial incident rays and 4.2 mrad for oblique rays with the polar angle of 0.1 rad. The maximum error in the scan patterns of the combination of two Risley prisms in the example were about 10 mrad in the x direction at the point of $\mathbf{s}_x = 0.2$ and $\mathbf{s}_y = 0.2$. Even though the errors tend to increase with the number of prisms, the first-order approximate formula will be useful for analyzing the scan patterns of Risley prisms.

References

1. C. Schwarze, "A new look at Risley prisms," *Photonics Spectra* **40**(6), 67–70 (2006).
2. V. Lavigne and B. Ricard, "Fast Risley prisms camera steering system: calibration and image distortions correction through the use of a three-dimensional refraction model," *Opt. Eng.* **46**, 043201 (2007).
3. G. Garcia-Torales, M. Strojnik, and G. Paez, "Risley prisms to control wave-front tilt and displacement in a vectorial shearing interferometer," *Appl. Opt.* **41**, 1380–1384 (2002).
4. C. T. Amirault and C. A. DiMarzio, "Precision pointing using a dual-wedge scanner," *Appl. Opt.* **24**, 1302–1308 (1985).
5. W. C. Stone, M. Juberts, N. Dagalakis, J. Stone, and J. Gorman, "Performance analysis of next-generation LADAR for manufacturing, construction, and mobility," *NISTIR 7117, National Institute of Standards and Technology* (2004).
6. B. D. Duncan, P. J. Bos, and V. Sergan, "Wide-angle achromatic prism beam steering for infrared countermeasures applications," *Opt. Eng.* **42**, 1038–1047 (2003).
7. P. J. Bos, H. Garcia, and V. Sergan, "Wide-angle achromatic prism beam steering for infrared countermeasures and imaging applications: solving the singularity problem in the two-prism design," *Opt. Eng.* **46**, 113001 (2007).
8. F. A. Jenkins and H. E. White, *Fundamentals of Optics*, 4th Ed., Sec. 2.7, McGraw-Hill Book Co., New York (1981).
9. F. A. Rosell, "Prism scanner," *J. Opt. Soc. Am.* **50**, 521–526 (1960).
10. M. Ostaszewski, S. Harford, N. Doughty, C. Hoffman, M. Sanchez, D. Gutow, and R. Pierce, "Risley prism beam pointer," *Proc. SPIE* **6304**, 630406 (2006).
11. G. F. Marshall, "Risley prism scan patterns," part of the *SPIE Conference on Optical Scanning: Design and Application*, Denver, Colorado, Vol. 3787, pp. 74–86 (1999).
12. W. L. Wolfe and G. J. Zissis, Eds., *The Infrared Handbook*, revised ed., Chapter 10, The Infrared Information Analysis (IRIA) Center, Environmental Research Institute of Michigan, Prepared for the Office of Naval Research, Department of the Navy, Washington, DC (1989).
13. Y. Li, "Third-order theory of the Risley-prism-based beam steering system," *Appl. Opt.* **50**, 679–686 (2011).
14. M. Born and E. Wolf, *Principles of Optics*, 7th Ed., Sec. 3.2.2, Cambridge University Press, Cambridge, UK (1999).
15. A. Ghaffar and Q. A. Naqvi, "Focusing of electromagnetic plane wave into uniaxial crystal by a three dimensional plano convex lens," *PIER* **83**, 25–42 (2008).
16. A. Ghaffar, A. A. Rizvi, and Q. A. Naqvi, "Fields in the focal space of symmetrical hyperboloidal focusing lens," *PIER* **89**, 255–273 (2009).
17. G. Arfken and H. J. Weber, *Mathematical Methods for Physicists*, 4th Ed., Sec. 3.3, Academic Press, San Diego, CA (1995).
18. E. Hecht, *Optics*, 4th Ed., Sec. 5.5.1, Addison Wesley, San Francisco, CA (2002).
19. H. S. Uhler, "Oblique deviation and refraction produced by prisms," *Am. Math. Monthly* **28**(1), 1–10 (1921).
20. R. E. Juhala and G. Dube, "Refractive beam steering," *Proc. SPIE* **5528**, 282–292 (2004).

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