Trade-offs between redundancy and increased rank for tomographic system matrices

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ABSTRACT

In this paper we examine whether increasing the rank of a tomographic system matrix by slightly changing the measurement lines might improve the reconstructed image quality in region-of-interest (ROI) tomography. The image quality is assessed by the size of the reconstructible region and by the reconstructed noise of the pixels in that region. A theorem is presented to specify cases when changing elements in the system matrix leads to solving new pixels. Numerical simulations were performed on several modest sized examples to measure the bias and variance of the reconstructed images using system matrices with the same size but different rank. The results show that while increasing the rank might help to solve new pixels, it can also lead to increases in the pixel variances of the initial ROI.

Keywords: region-of-interest tomography, tomographic system matrix, matrix rank

1. INTRODUCTION

Region-of-interest (ROI) tomography consists of reconstructing only a part of the whole image from truncated data.[1] Truncated data arises when the patient is not completely inside the field of view (FOV). ROI tomography can be useful to reduce the radiation dose given to the patient or for partial reconstruction when the patient is larger than the FOV. The reconstructible ROI depends on the acquired projection lines, described in the system matrix mapping the image to the measurements.

We investigate how small modifications of the acquired projection lines might improve ROI reconstruction. One improvement could be the size of the ROI, i.e. the number of reconstructible pixels. Another improvement might be in the reconstruction noise, as described by the pixel variances in the ROI. Our previous work[2] has shown that a tiny change in the system matrix elements might lead to a large improvement of the variance in the ROI when this change causes the rank of the system matrix to drop. Those results were in line with a previous theorem about the the impact of new data on the reconstruction quality.[3] In this paper, we derive a theorem specifying how changes in the system matrix can increase the size of the ROI, and show that a trade-off must be made between the size of the ROI and the reconstruction noise.

2. MATERIALS AND METHODS

2.1 Theory

Let us consider the $m \times n$ system matrix S defined such that $rank(S) < n$. The linear system

$$y = Sx \tag{1}$$

with $x \in \mathbb{R}^n$ the image vector and $y \in \mathbb{R}^m$ the measurements vector, does not have a unique solution, although some components of x (some pixels) might be uniquely determined. Those elements correspond to the ROI pixels. In this model, S_{ij} represents the length of the intersection of line i with pixel j, assuming unit pixels. Note that there might be linear dependencies between the rows of S, i.e. the rank of S could be smaller than m. Now we consider the $m \times n$ matrix C defining the linear system

$$z = Cx \tag{2}$$

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(for the same x as in equation (1)), such that S and C are equal except for one element. Consequently,

$$\text{rank}(S) - 1 \leq \text{rank}(C) \leq \text{rank}(S) + 1 \tag{3}$$

i.e. changing one element in S might increase the rank by one, decrease it by one, or not change the rank. We establish the following theorem

Theorem. Let S be a $m \times n$ matrix. If there exists a $m \times n$ matrix C such that $C_{ij} = S_{ij}$ for all $(i, j) \neq (a, b)$ and $\text{rank}(C) = \text{rank}(S) + 1$, then (i) C can uniquely determine at least the b^{th} component of x, and (ii) S cannot uniquely determine the b^{th} component of x.

Note that C might also have a non-trivial nullspace, (i.e. $\text{rank}(C) < n$), but x_b would nevertheless be solvable. Note also, that only one element of S is allowed to change. The example in the appendix shows that the theorem fails if the rank increases by one, but that S and C differ by more than one element.

Proof. (i) We first note that the change in rank means that S and C cannot be the same matrix, and therefore $S_{ab} \neq C_{ab}$. We will show that there exist scalars $\{\beta_i\}_{i=1,2...m}$ such that

$$x_b = \sum_{i=1}^{n} \beta_i z_i \tag{4}$$

and thus x_b, can be uniquely determined from (2).

The only difference between y and z is the a^{th} measurement, given by

$$y_a = \sum_{j=1}^{n} S_{aj} x_j \tag{5}$$

for y and

$$\begin{aligned} z_a &= \sum_{j=1}^{n} C_{aj} x_j \\ &= \sum_{\substack{j=1 \\ j \neq b}}^{n} S_{aj} x_j + C_{ab} x_b \end{aligned} \tag{6}$$

for z. Subtracting (6) from (5) gives

$$y_a - z_a = (S_{ab} - C_{ab}) x_b \tag{7}$$

On the other hand, since S and C are the same size, if $\text{rank}(C) = \text{rank}(S) + 1$, then there is a linear dependency between the rows of S. More specifically, if a change only in row a increases the rank, then that row is a linear combination of the other rows. (Recall that the rank of a matrix equals the dimension of the row-space.) Writing $S^{(i)}$ for the i^{th} row of S, we have, for some $m - 1$ scalars $\{\alpha_i\}_{i=1,2...m; i \neq a}$,

$$S^{(a)} = \sum_{\substack{i=1 \\ i \neq a}}^{m} \alpha_i S^{(i)} \tag{8}$$

and since $y_i = x \cdot S^{(i)}$ we see that

$$\begin{aligned} y_a = x \cdot S^{(a)} = \sum_{\substack{i=1 \\ i \neq a}}^{m} \alpha_i x \cdot S^{(i)} = \sum_{\substack{i=1 \\ i \neq a}}^{m} \alpha_i x \cdot C^{(i)} \\ = \sum_{\substack{i=1 \\ i \neq a}}^{m} \alpha_i z_i \end{aligned} \tag{9}$$

So, the difference between the two measurements is

$$y_a - z_a = \sum_{\substack{i=1 \\ i \neq a}}^{m} \alpha_i z_i - z_a \tag{10}$$

Combining (7) and (10) gives

$$x_b = \frac{1}{S_{ab} - C_{ab}} \Big(\sum_{\substack{i=1 \\ i \neq a}}^{m} \alpha_i z_i - z_a \Big) = \sum_{i=1}^{m} \beta_i z_i \tag{11}$$

where the β_i are defined by the right hand equality of (11) since $S_{ab} - C_{ab} \neq 0$. Therefore, the component x_b can be determined from the z_i measurements.

(ii) We will prove the contrapositive: if x_b can be found using (1) then changing S to C will not increase the rank.

We define $e_b \in \mathbb{R}^n$ to be all zeros, except the b^{th} component, where it is one. Thus $x \cdot e_b = x_b$. Now since x_b can be solved from (1), there must exist scalars $\{\gamma_i\}$ such that

$$\begin{aligned} x_b &= \sum_{i=1}^{m} \gamma_i y_i = \sum_{i=1}^{m} \gamma_i (x \cdot S^{(i)}) \\ &= x \cdot (\sum_{i=1}^{m} \gamma_i S^{(i)}) \end{aligned} \tag{12}$$

and since (12) holds for all $x \in \mathbb{R}^n$, we must have

$$e_b = \sum_{i=1}^{m} \gamma_i S^{(i)} \tag{13}$$

which equivalently says that row-reducing the matrix S will result in one of the rows becoming e_b.

We now observe that $C^{(a)}$ is a linear combination of $S^{(a)}$ and e_b, and note that e_b is a linear combination of the rows of S (see (13)). Every row of C can therefore be obtained from a linear combination of the rows of S, so the rank of C cannot exceed the rank of S (the dimension of the row-space of C cannot exceed the dimension of the row-space of S).

We have shown that, if x_b is solvable from S (and S and C are the same matrix except for element C_{ab}) then C cannot have higher rank than S. Equivalently, if rank(C) = rank$(S) + 1$, then x_b cannot be uniquely determined from S. □

Corollary. If C is such that $C_{ij} = S_{ij}$ for $(i, j) \neq (a_k, b_k)$ for $k \in \{1...K\}$ and rank(C) = rank$(S) + K$, then C can uniquely determine at least the K b_k components of x, and these K components were *not* uniquely solvable using S. Furthermore the K matrix elements that changed all lie on different rows and different columns ($a_k \neq a_l$ and $b_k \neq b_l$ for $k, l \in \{1 \ldots K\}$).

Proof. For the rank to increase by K, at least K rows of the matrix must have changed. But with only K changes of the matrix, each changed element must have been on a different row. Similarly, since the rank is also equal to the dimension of the column space, the same argument applies to the columns, and each changed element is in a different column. The corollary is now obtained by successively applying the theorem. We recall that changing one element of a matrix can cause the rank to change by one at most, i.e. if $K = 1$, rank$(C) \leq$ rank$(S) + 1$. Therefore, as K changes in the matrix cause rank(C) = rank$(S) + K$, each changed element (a_k, b_k) must have increased the rank by 1, and, according to the theorem, lets us determine the b_k component of x, and b_k was not solvable before the change. □

Remark. It is easily seen that if S can uniquely determine some components of x, then C can also determine those same components, as well as the additional component(s). In short, under the hypotheses of the theorem, matrix C can always determine more components than S can.

In the following sections, we illustrate the theorem and the corollary with several examples.

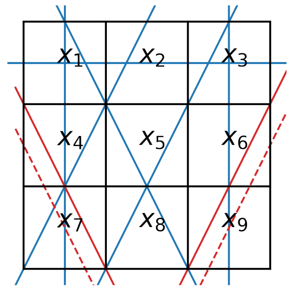

Figure 1. Projection lines corresponding to the system matrix S. The red lines correspond to the 7^{th} and 8^{th} rows, and the dotted lines to the new measurements after changing one element in each row.

2.2 Small-scale examples

We define the following 8×9 system matrix

$$S = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & A & 0 & A & 0 & 0 & A & 0 & 0 \\ A & 0 & 0 & 0 & A & 0 & 0 & A & 0 \\ 0 & 0 & A & 0 & A & 0 & 0 & A & 0 \\ 0 & 0 & 0 & A & 0 & 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A & 0 & 0 & A \end{pmatrix} \tag{14}$$

with $A = \sqrt{5}/2$, which allows exact reconstruction of the first three elements of x (almost the same matrix was used in Ref. 2). The 8 corresponding projection lines are shown in Fig. 1. The matrix S is rank deficient with $\text{rank}(S) = 6$. Now we define a matrix C such that

$$\begin{cases} C_{ij} = A/2 & \text{if } (i,j) = (8,6) \\ C_{ij} = S_{ij} & \text{otherwise.} \end{cases} \tag{15}$$

and $\text{rank}(C) = 7$. This change corresponds to a downwards half pixel translation of the eighth projection line (see Fig. 1).

We took a $n \times 1$ image vector x with values between 0 and 10, and simulated noiseless measurements according to equations (1) and (2). The values in the 3×3 image were

$$x = \begin{pmatrix} 5.75 & 7.43 & 0.50 \\ 9.88 & 3.57 & 8.79 \\ 0.37 & 1.24 & 6.32 \end{pmatrix}. \tag{16}$$

The estimated image $\hat{x}$ was reconstructed by applying the pseudo-inverse of the system matrices. The absolute bias

$$|\hat{x}_j - x_j| \tag{17}$$

for each reconstruction was computed in order to establish which pixels x_j were correctly determined, a bias of zero corresponding to a correct reconstruction. The results, shown in Table 1 confirmed that, as expected, pixels $\{x_1, x_2, x_3\}$ can be reconstructed from both S and C. Also, the matrix C uniquely determines pixel x_6 as predicted by the theorem, given that the only different element between S and C was $(8, 6)$. In addition, pixel x_9 was fortuitously reconstructed exactly, since the eighth projection line only crossed pixels x_6 and x_9.

In order to evaluate the variance of the estimated pixels, the measurements were repeated 10^5 times with additional Gaussian noise of variance $\sigma^2 = 1$. In Table 1, it is observed that the variance of pixels x_1 to x_3, which are the only ones correctly reconstructed by both S and C, increased when C was used for reconstruction.

Table 1. Bias and variance of the estimated solution using either S or C.

	S		C (Eq. 15)		C (Eq. 18)	
	Bias	Var.	Bias	Var.	Bias	Var.
Pixel 1	$< 10^{-14}$	0.47	$< 10^{-15}$	0.47	$< 10^{-14}$	0.49
Pixel 2	$< 10^{-14}$	0.80	$< 10^{-14}$	0.89	$< 10^{-14}$	1.08
Pixel 3	$< 10^{-15}$	0.62	$< 10^{-14}$	0.93	$< 10^{-14}$	1.02
Pixel 4	4.75	N/R	4.75	N/R	$< 10^{-14}$	6.45
Pixel 5	1.16	N/R	1.16	N/R	1.16	N/R
Pixel 6	1.23	N/R	$< 10^{-16}$	10.9	$< 10^{-13}$	11.35
Pixel 7	4.75	N/R	4.75	N/R	$< 10^{-13}$	3.99
Pixel 8	1.16	N/R	1.16	N/R	1.16	N/R
Pixel 9	1.23	N/R	$< 10^{-16}$	5.14	$< 10^{-16}$	5.25

The increase was by as much as 50% for pixel x_3. The variances of the new reconstructible pixels, x_6 and x_9, were larger with C, but their previous variances were not relevant since they were not accurately reconstructible anyway (the pseudo-inversion produced a large bias).

In order to illustrate the corollary, we performed the same experiment with C defined as

$$\begin{cases} C_{ij} = A/2 & \text{if } (i,j) = \{(7,4),(8,6)\} \\ C_{ij} = S_{ij} & \text{otherwise.} \end{cases} \tag{18}$$

resulting in $\text{rank}(C) = 8$ (see Fig. 1).

According to the corollary, the matrix C should be able to determine pixels x_4 and x_6, which was confirmed by the zero bias for these two pixels in Table 1. Two additional pixels, x_7 and x_9, were also accurately reconstructed. On the other hand, the variances for pixels $\{x_1, x_2, x_3\}$ have increased even more (by as much as 65% for pixel x_3).

2.3 Medium scale example

We performed the same experiment but for a larger 32×32 image of the Shepp Logan phantom (Fig. 2). We built a 396×1024 system matrix S of rank 369 which was able to accurately reconstruct a 16×16 ROI in the upper left corner of the image. Similar to the small-scale example, we built the matrix C by changing K elements (one per row). We chose $K = 3$, such that $\text{rank}(C) = 369 + 3 = 372$. Specifically, the changes occurred in columns 257, 272, and 513. The three projection line that were changed in S are shown in Fig. 2.

The bias maps between the reconstructions (without simulated noise) using either S or C and the reference phantom are shown in Fig. 3. The pixels in the upper left corner of the reconstructed image using S appear in blue, corresponding to a very small bias, hence they are accurately reconstructed. As expected, the 3 pixels

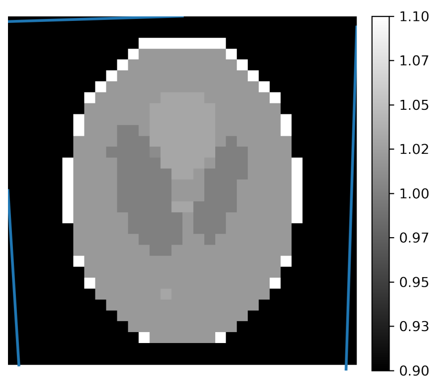

Figure 2. Shepp-Logan phantom. The three blue lines correspond to the projection lines in S that were changed.

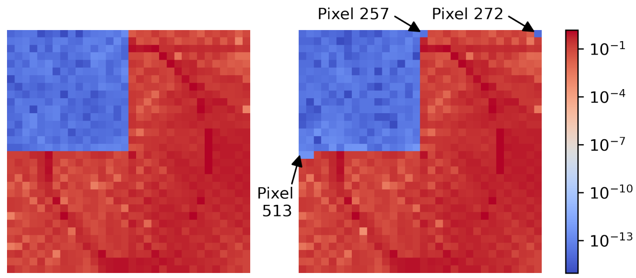

Figure 3. Absolute bias between the reconstructed image using S (left) or C (right) and the reference Shepp Logan phantom. Note the logarithmic scale.

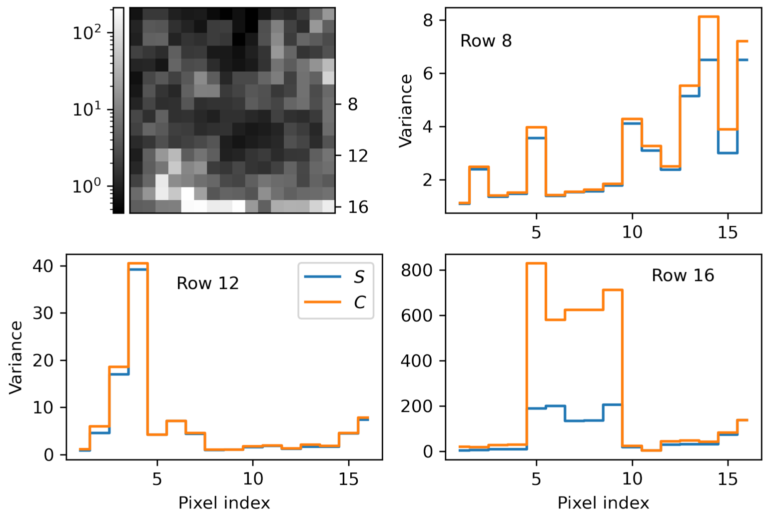

Figure 4. Top left: variance in the 16×16 ROI reconstructed with S. Top right, bottom left, bottom right: profiles through the 8^{th}, 12^{th}, and 16^{th} row of the variance map using either S or C.

corresponding to the changed columns 257, 272, and 513, were also accurately reconstructed with matrix C. Additionally, figure 3 shows that pixel 514 (to the right of pixel 513) was also accurately reconstrued using the matrix C.

The differences between the variances of the pixels in the upper left 16×16 ROI reconstructed using either S or C are shown in Fig. 4. For all pixels in the ROI, the variance was larger when C was used to reconstruct the image.

3. DISCUSSION

Our results have shown that, while increasing the rank of a matrix by changing its elements might increase the number of reconstructible pixels, it can also lead to a larger variance in the initial ROI. Therefore, a trade-off must be made between the size of the ROI and the reconstruction noise. For an $m \times n$ matrix, increasing the rank means increasing the number of independent equations and potentially solving a new variable. On the other hand, a rank deficient matrix contains redundant information, i.e. a possible smaller variance for the variables that can be solved.

The proposed theorem specifies when changes in the matrix will solve new variables, but is limited to one change per row. In practice, only projection lines close to the edge of the image can be modified since only one value in the system matrix is allowed to change. In our examples, the K new reconstructible pixels predicted by the theorem were all at the edges of the image, although other pixels can also be reconstructed. Nevertheless, increasing the rank of a matrix (while keeping its size constant) will increase the chance of solving new variables, while reducing the number of redundant measurements that might have been used to reconstruct ROI pixels.

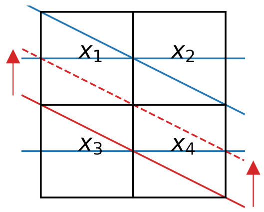

Figure 5. Projection lines corresponding to the 4×4 system matrices. The full red line represents the last row of S and the dotted line the last row of C.

4. CONCLUSION

A theorem specifying when a change in a tomographic system matrix can help solve new pixels has been proven. The study of the bias and the variance of the reconstructed images has shown that the additional reconstructible pixels might come at the cost of a reduced variance in the original ROI.

APPENDIX A.

Here we show that the hypothesis of the theorem, that only one element must differ between S and C, cannot be relaxed for the conclusions of the theorem to hold. We give an example of matrices S and C of size 4×4 such that $\text{rank}(S) = 2$ and $\text{rank}(C) = 3$ but no component of x can be uniquely reconstructed using either S or C. Two of the matrix elements of C differ from those of S. This example illustrates that, to apply the theorem, only one element of the matrix is allowed to change, when the rank increases by 1.

The example corresponds to a very simple tomographic system with x representing a 2×2 image. Using the same value of $A = \sqrt{5}/2$, we define S as

$$S = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ A & A & 0 & 0 \\ 0 & 0 & A & A \end{pmatrix} \tag{19}$$

and for the matrix C the last row is replaced by $(A\ 0\ 0\ A)$. The system is shown in Figure 5.

It is easy to see that $\text{rank}(S) = 2$ and $\text{rank}(C) = 3$ and that no component of x can be determined using matrix S. For the matrix C, the one-dimensional nullspace is spanned by the vector $(1, -1, 1, -1)$ and therefore no component of x can be uniquely resolved when using the matrix C.

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REFERENCES

[1] R. Clackdoyle and M. Defrise, "Tomographic reconstruction in the 21st century," *IEEE Signal Processing Magazine*, vol. 27, no. 4, pp. 60-80, Jul. 2010.

[2] F. Khellaf, R. Clackdoyle, S. Rit and L. Desbat, "Tiny changes in tomographic system matrices can cause large changes in reconstruction quality", *Phys. Med. Biol.*, 2022.

[3] M. Defrise, R. Clackdoyle, L. Desbat, F. Noo, and J. Nuyts, "Do additional data improve region-of-interest reconstruction?" *The 16th International Meeting on Fully Three-Dimensional Image Reconstruction in Radiology and Nuclear Medicine*, Leuven, Belgium. July 19-23, 2021. Pages: 38-43.