Study on the complex dynamics and chaos control for Bertrand duopoly game models with emission charges

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ABSTRACT

In this paper we consider a nonlinear Bertrand game model and investigate the dynamical properties. It is assumed that the two competitors follow linear cost functions and quadratic emission charges. We construct a two-dimensional system for the model first. Then we examine the equilibrium points and their stability conditions. The chaotic properties are presented numerically via bifurcation diagrams, maximum Lyapunov exponent, sensitive dependence on initial conditions and strange attractors. Finally, we control the chaotic behavior by time-delay feedback method.

Keywords: Emission charges, Bertrand duopoly model, heterogeneous expectations, bifurcation, chaos control

1. INTRODUCTION

In recent years, the research on environmental policies for controlling environmental pollution has received increasing attention. Shahzad[1] analyzed empirical and theoretical literature up to 2020, and the results showed that energy used in economic activities had a positive impact on pollutant release. However, the role of environmental taxation is still unclear. In this context, some literature has conducted theoretical analysis from a new perspective of nonlinear dynamics on whether environmental taxation controls the fate of polluting enterprises.

The early articles on the role of environmental taxes mainly focused on complete competition and market disruption, as seen in Buchanan[2] and Barnett[3]. The study has been extended to the oligopoly market in recent years. The impact of emissions taxes under fixed quantity competition and endogenous structure is investigated by Katsoulacos and Xepapadeas[4]. The effects of R&D investments under different environmental policies are analyzed by Montero[5]. The existence of Nash equilibrium points when firms' payoffs consist of pollution abatement R&D and pollution control for exceeding the emission standards are demonstrated by Okuguchi and Szidarovszky[6]. The effect of changes in environmental charges on the concentration of nonpoint source pollution in n markets of firms with differentiated products is investigated by Matsumoto et al.[7]. A theoretic method is proposed by Matsumoto and Szidarovszky[8]and they used a hyperbolic price function, whereby they investigated the effect of environmental changes on aggregate and individual nonpoint source pollution. The effect of emissions charges is considered by Mamada and Perrings[9] in market structure, showing the stability conditions for duopoly and monopoly markets. Naimzadal and Pireddu[10] extended the homogeneous product case of Reference[9] to the more general differentiated goods.

In the present paper, we investigate the dynamical properties of a Bertrand duopoly with emission charges and differentiated goods, in which each firm behaves with different expectation. The rest of the paper can be briefly described as follows. The model describing the game is given in Section 2. The two-dimensional system is analyzed in Section 3. Section 4 presents a numerical example to demonstrate the various dynamic behaviors of the system. Section 5 analyzes the chaos control problem by time-delay feedback technique. The final conclusion is presented in Section 6.

2. MODEL DISCRIPTION

We investigate the dynamical properties of a Bertrand duopoly model, in which the two firms produce differentiated goods. The decisions are made at discrete points $t = 0,1,2,\cdots$. At time t, each competitor need take an expectation of the rival's price decision in the following $t+1$ time. Assume that q_1, q_2 are the production quantities of the two competitors. The utility function of the representative consumer is denoted by:

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$$U(q_1,q_2)=\alpha(q_1+q_2)-\frac{1}{2}(q_1{}^2+q_2{}^2+2dq_1q_2) \tag{1}$$

in which $\alpha(\alpha>0)$ denotes the market size and $d\in(0,1)$ measures the degree of the production. A larger degree of product differentiation implies a smaller d. From (1), we can derive the inverse demand functions as

$$p_1(q_1,q_2)=\alpha-q_1-dq_2 \quad \text{and} \quad p_2(q_1,q_2)=\alpha-q_2-dq_1 \tag{2}$$

By (2), we can get the direct demand functions as

$$q_1(p_1,p_2)=\frac{\alpha(1-d)-p_1+dp_2}{1-d^2} \quad \text{and} \quad q_2(p_1,p_2)=\frac{\alpha(1-d)-p_2+dp_1}{1-d^2} \tag{3}$$

At each time t, emissions of firm $i\in\{1,2\}$ are given by:

$$u_{i,t}=\varepsilon q_{i,t} \tag{4}$$

where $\varepsilon(\varepsilon>0)$ represents emissions per unit of output.

Emission charges of Firm $i\in\{1,2\}$ at time t are quadratic in the level of emission as:

$$C_{i,t}^{e}=bu_{i,t}+\frac{1}{2}mu_{i,t}{}^2 \tag{5}$$

where $b>0$ and $m>0$.

The cost functions are assumed to be linear:

$$C_i(q_i)=cq_i \tag{6}$$

where $i=1,2$ and $c(c>0)$ denotes the marginal production cost.

Under these assumptions, we can compute the profits of the two competitors as follows:

$$\begin{aligned} P_1(p_1,p_2)&=p_1q_1-cq_1-c_{1,t}^{e}\\ &=\frac{1}{(1-d^2)^2}\left[\alpha(1-d)-p_1+dp_2\right]\left[(1-d^2+\frac{1}{2}m\varepsilon^2)p_1-\frac{1}{2}md\varepsilon^2p_2+A\right]\\ P_2(p_1,p_2)&=p_2q_2-cq_2-c_{2,t}^{e}\\ &=\frac{1}{(1-d^2)^2}\left[\alpha(1-d)-p_2+dp_1\right]\left[(1-d^2+\frac{1}{2}m\varepsilon^2)p_2-\frac{1}{2}md\varepsilon^2p_1+A\right] \end{aligned} \tag{7}$$

where

$$A=-(c+b\varepsilon)(1-d^2)-\frac{1}{2}m\varepsilon^2\alpha(1-d)$$

From (7), we can find the marginal profits as:

$$\begin{aligned} \frac{dP_1}{dp_1}&=\frac{1}{(1-d^2)^2}\left[(2d^2-m\varepsilon^2-2)p_1+d(1-d^2+m\varepsilon^2)p_2+B\right]\\ \frac{dP_2}{dp_2}&=\frac{1}{(1-d^2)^2}\left[(2d^2-m\varepsilon^2-2)p_2+d(1-d^2+m\varepsilon^2)p_1+B\right] \end{aligned} \tag{8}$$

where

$$B=(c+b\varepsilon)(1-d^2)+\alpha(1-d)(1-d^2+m\varepsilon^2)>0$$

The first competitor is assumed to be abounded rational player, i.e., it modifies its price decisions dynamically according to the marginal profits. Let $k>0$, the first competitor's dynamical equation is:

$$\frac{p_1(t+1)-p_1(t)}{p_1(t)}=k\frac{dP_1}{dp_1} \tag{9}$$

where k denotes a positive parameter that represents the speed of price adjustment. Relative variations of the price are proportional to the marginal profit are captured by it. The second competitor is a naive player:

$$p_2(t+1)=\frac{d(1-d^2+m\varepsilon^2)p_1(t)+B}{2+m\varepsilon^2-2d^2} \tag{10}$$

Therefore, we obtain the two-dimensional system of the price game as follows:

$$\begin{cases} p_1(t+1)=p_1(t)+kp_1(t)\dfrac{dP_1}{dp_1} \\ p_2(t+1)=\dfrac{d(1-d^2+m\varepsilon^2)p_1(t)+B}{2+m\varepsilon^2-2d^2} \end{cases} \tag{11}$$

3. EQUILIBRIUM POINTS AND LOCAL STABILITIES

For the dynamical system (11), equilibriums can be achieved by solving the following algebraic equations:

$$\begin{cases} k\cdot p_1^*\cdot\dfrac{1}{(1-d^2)^2}\left[\left(2d^2-m\varepsilon^2-2\right)p_1^*+d\left(1-d^2+m\varepsilon^2\right)p_2^*+B\right]=0 \\ p_2^*=\dfrac{d\left(1-d^2+m\varepsilon^2\right)p_1^*+B}{2+m\varepsilon^2-2d^2} \end{cases} \tag{12}$$

by letting $p_1\left(t+1\right)=p_1(t)=p_1^*$ and $p_2\left(t+1\right)=p_2(t)=p_2^*$, we obtain:

- If $p_1^*=0$, then $p_2^*=\dfrac{B}{2+m\varepsilon^2-2d^2}$ and we get the boundary equilibrium point:

$$E_0=\left(0,\frac{B}{2+m\varepsilon^2-2d^2}\right) \tag{13}$$

- If $\dfrac{dP_1}{dp_1}=\dfrac{dP_2}{dp_2}=0$, then we form the system:

$$\begin{cases} \left(2d^2-m\varepsilon^2-2\right)p_1^*+d\left(1-d^2+m\varepsilon^2\right)p_2^*+B=0 \\ \left(2d^2-m\varepsilon^2-2\right)p_2^*+d\left(1-d^2+m\varepsilon^2\right)p_1^*+B=0 \end{cases} \tag{14}$$

The solutions are $p_1^*=p_2^*=\dfrac{-B}{D}$, where $D=d\left(1-d^2+m\varepsilon^2\right)+\left(2d^2-m\varepsilon^2-2\right)$. We can easily discern that $D<0$ and give the Nash equilibrium point:

$$E_*=\left(\frac{-B}{D},\frac{-B}{D}\right) \tag{15}$$

For an evaluation of the local stability at the equilibrium points, the Jacobian matrix of system (11) is computed as:

$$J(p_1,p_2)=\begin{pmatrix} 1+\frac{k}{(1-d^2)^2}\left[2\left(2d^2-m\varepsilon^2-2\right)p_1+d\left(1-d^2+m\varepsilon^2\right)p_2+B\right] & \frac{kd\left(1-d^2+m\varepsilon^2\right)p_1}{(1-d^2)^2} \\ \frac{d(1-d^2+m\varepsilon^2)}{2+m\varepsilon^2-2d^2} & 0 \end{pmatrix} \quad (16)$$

Theorem 1. The boundary equilibrium point E_0 is unstable.

Proof. Submitting E_0 into (16) gives:

$$J(E_0)=\begin{pmatrix} 1+\frac{kB}{(1-d^2)^2}\left[\frac{d(1-d^2+m\varepsilon^2)}{2+m\varepsilon^2-2d^2}+1\right] & 0 \\ \frac{d(1-d^2+m\varepsilon^2)}{2+m\varepsilon^2-2d^2} & 0 \end{pmatrix} \quad (17)$$

The eigenvalues of $J(E_0)$ are given by:

$$\lambda_1=0 \text{ and } \lambda_2=1+\frac{kB}{(1-d^2)^2}\left[\frac{d(1-d^2+m\varepsilon^2)}{2+m\varepsilon^2-2d^2}+1\right] \quad (18)$$

Since $|\lambda_2|>1$, so the point E_0 is unstable.

Theorem 2. The Nash equilibrium point E_* is asymptotically locally stable if

$$k<\frac{2(2+m\varepsilon^2-2d^2)(1-d^2)^2}{p^*[(2+m\varepsilon^2-2d^2)^2+d^2(1-d^2+m\varepsilon^2)^2]}$$

with

$$p^*=\frac{-B}{D}$$

Proof. Submitting E_* into (16) leads to:

$$J(E_*)=\begin{pmatrix} 1-\frac{k(2+m\varepsilon^2-2d^2)p^*}{(1-d^2)^2} & \frac{kd(1-d^2+m\varepsilon^2)p^*}{(1-d^2)^2} \\ \frac{d(1-d^2+m\varepsilon^2)}{2+m\varepsilon^2-2d^2} & 0 \end{pmatrix} \quad (19)$$

The trace (Tr) and the determinant (Det) of the Jacobian matrix $J(E_*)$ are the decisive factors for the stability of the equilibrium in the discrete time system. The trace and determinant can be computed as follows:

$$Tr=1-\frac{k(2+m\varepsilon^2-2d^2)p^*}{(1-d^2)^2} \text{ and } Det=-\frac{kd^2(1-d^2+m\varepsilon^2)^2p^*}{(2+m\varepsilon^2-2d^2)(1-d^2)^2} \quad (20)$$

It is well known that asymptotic stability of the Nash equilibrium is determined by the following conditions:

$$\begin{aligned} &(i)1-Det>0 \\ &(ii)1-Tr+Det>0 \\ &(iii)1+Tr+Det>0 \end{aligned} \quad (21)$$

The conditions (i) and (ii) are, respectively, equivalent to:

$$\begin{cases} 1 - Det = 1 + \dfrac{kd^2(1-d^2+m\varepsilon^2)^2 p^*}{(2+m\varepsilon^2-2d^2)(1-d^2)^2} > 0 \\ 1 - Tr + Det = kp^* \dfrac{(2+m\varepsilon^2-2d^2)^2 - d^2(1-d^2+m\varepsilon^2)^2}{(2+m\varepsilon^2-2d^2)(1-d^2)^2} > 0 \end{cases} \tag{22}$$

It is clear to see that these two inequalities are always satisfied. On the other hand, the third condition can be rewritten as:

$$2(2+m\varepsilon^2-2d^2)(1-d^2)^2 - kp^*[(2+m\varepsilon^2-2d^2)^2 + d^2(1-d^2+m\varepsilon^2)^2] > 0 \tag{23}$$

which can be further reduced to:

$$k < \frac{2(2+m\varepsilon^2-2d^2)(1-d^2)^2}{p^*[(2+m\varepsilon^2-2d^2)^2 + d^2(1-d^2+m\varepsilon^2)^2]} \tag{24}$$

4. NUMERICAL EXPERIMENT

This part provides some numerical experimentation to explain the complex dynamical properties of systems (11). The complex dynamic behaviors also exhibit significant differences under different parameters. In this simulation, we set $a=5$, $d=0.5$, $c=1$, $\varepsilon=0.2$, $b=0.2$, $m=0.5$.

With respect to parameter k, figures 1 and 2 respectively display the bifurcation diagram and the associated maximum Lyapunov exponents of system (11). They show the Nash equilibrium point is basically stable when $k \in (0, 0.29)$ and two branches appear when $k = 0.29$. As k becomes larger, system (11) loses stability, exhibiting complex dynamics such as period bifurcations and chaotic behaviors.

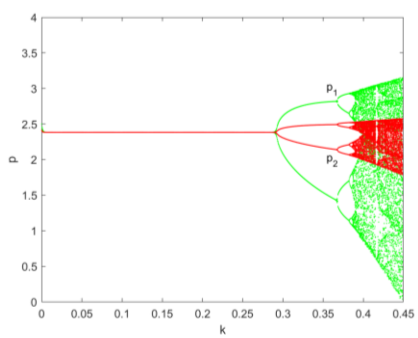

Figure 1. Bifurcation diagram of k.

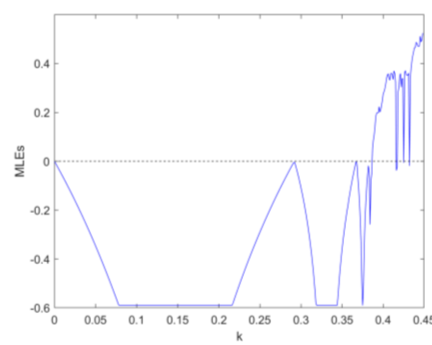

Figure 2. Maximum Lyapunov exponent.

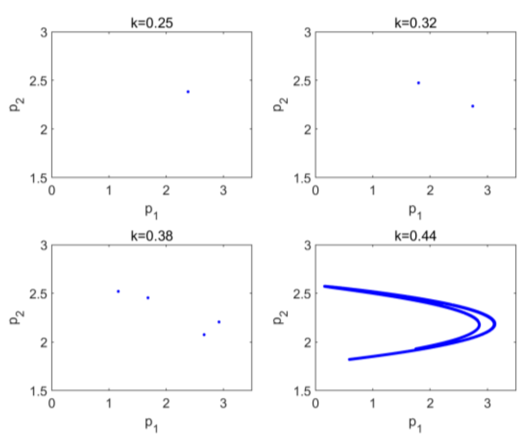

Figure 3. Phase diagrams of Figures 1 and 2 with a spectrum of k values.

Figure 3 presents the two-dimensional chaotic attractors, revealing a fractal structure for system (11). Around the Nash equilibrium point, there is a generation of more complex bounded attractors, identified as high order periodic cycles or chaotic attractors.

One of the key elements of chaotic dynamics is sensitive dependence on initial conditions. Figure 4 shows the orbits of two Firms' price. The initial prices are $(p_1^1(0), p_2^1(0)) = (2.5, 2.5)$ and $(p_1^2(0), p_2^2(0)) = (2.5001, 2.5)$. In the figure, the blue line represents the price of Firm 1 and the red color line is the price of Firm 2. It can be clearly seen that after a series of iterations, even if the initial price of a firm changes slightly, the disparity between them accumulates swiftly.

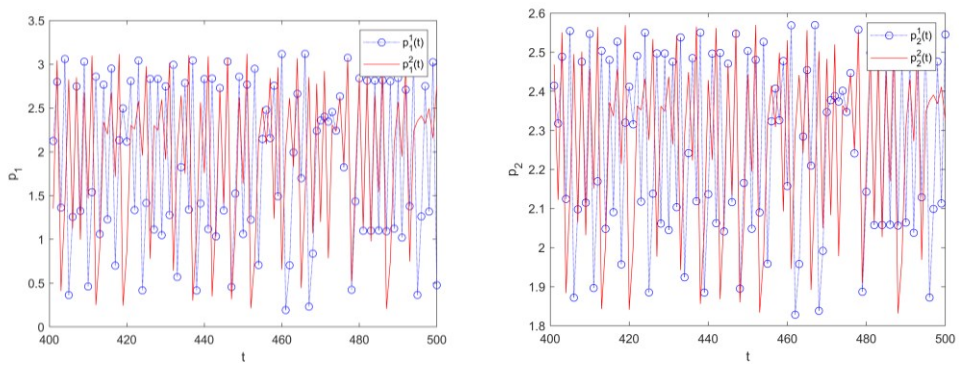

Figure 4. Sensitive dependence on initial conditions for system (11) in the time periods [400,500] when $k = 0.44$.

5. CHAOS CONTROL

In this section, we impose controls on the dynamic system (11). We apply time-delay feedback control method to manage the chaotic dynamics for our price duopoly model. To this end, the first equation of system (11) is adjusted by appending the management term $v(p_1(t) - p_1(t+1))$, where $v > 0$ denotes a control parameter. By properly adjusting v, the chaos of the system will be significantly reduced or eliminated. The managed system is:

$$\begin{cases} p_1(t+1) = p_1(t) + kp_1(t)\dfrac{dP_1}{dp_1} + v(p_1(t) - p_1(t+1)) \\ p_2(t+1) = \dfrac{d(1-d^2+m\varepsilon^2)p_1(t) + B}{2+m\varepsilon^2-2d^2} \end{cases} \tag{25}$$

Jacobian matrix of the controlled system (25) at Nash equilibrium point can be computed as:

$$J_2(E_*) = \begin{pmatrix} 1 - \dfrac{k(2+m\varepsilon^2-2d^2)p^*}{(1+v)(1-d^2)^2} & \dfrac{kd(1-d^2+m\varepsilon^2)p^*}{(1+v)(1-d^2)^2} \\ \dfrac{d(1-d^2+m\varepsilon^2)}{2+m\varepsilon^2-2d^2} & 0 \end{pmatrix} \tag{26}$$

With the parameters set to $a = 5$, $d = 0.5$, $c = 1$, $\varepsilon = 0.2$, $b = 0.2$, $m = 0.5$, $k = 0.44$, the condition that the controlled system (25) is stable is $v > 0.52$.

In Figure 5, the system (25) gradually stabilizes from chaos as the control parameter v increases. When $v > 0.52$, the system (25) eventually stabilizes. Figure 6 displays the evolutionary trajectory of the control system (25) for $v = 0.55$, $v = 0.59$, and $v = 0.63$ with other parameters held invariant. We see that the time-delayed feedback control technique can be effectively applied to switch from chaotic trajectories to an equilibrium status, which helps that the firms adjust strategies and guarantees that the market evolves in a structured manner.

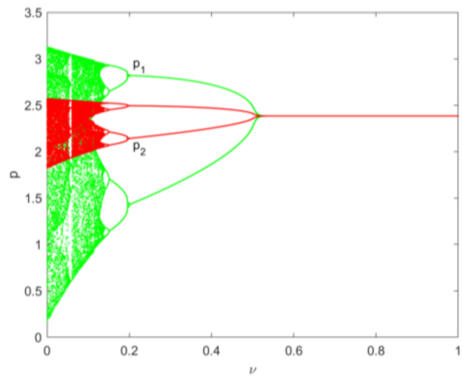

Figure 5. Bifurcation diagram of ν .

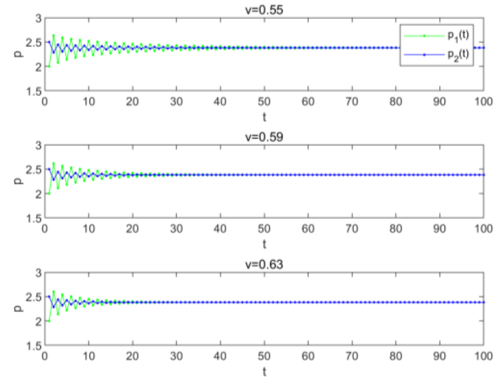

Figure 6. Evolutions with various values of ν .

6. CONCLUSION

In this paper we construct a price competition model, where the cost function is linear and the emission charge is quadratic. We investigate the dynamical properties of the system under different expected conditions. It is found that the boundary equilibrium point is unstable and the Nash equilibrium point is locally stable. The system will be in a chaotic state when the price adjustment parameter of the firm is higher than a certain critical value. Numerical simulations are used to show the complexity in the evolution of the dynamical system. Finally, we have also demonstrated that the method of feedback control with time delay can be utilized to restore the system to its stable condition from a state of chaos.

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