Numerical solution of Volterra integral equation with piecewise intervals by iterative method

Keyan Wang*
School of Mathematics and Statistics, Hengyang Normal University, Hengyang 421008, Hunan, China

ABSTRACT

In this paper, an iterative method is introduced to approximate the solution of nonlinear Volterra integral equation with piecewise intervals. Firstly, we transform the nonlinear integral equation into a system of nonlinear algebraic equations and then proceed to solve these equations. We consider certain conditions of the integral equation and study the convergence analysis of iterative method. A numerical example is provided to confirm the reliability and accuracy of the proposed method.

Keywords: Volterra integral equation, piecewise intervals, iterative method

1. INTRODUCTION

In this paper, let $I=[a,b]$ be an interval on the real zxis, we consider the following integral problem: find $u(x)$ such that

$$u(x)=f(x)+\sum_{p=1}^{M}\lambda_p\int_{a_p}^{x}k_p(x,y,u(y))dy, \tag{1}$$

where $x,y\in I$, $a\le a_p\le b$, $\lambda_p\in R$, R is the set of real number, f and k_p are known smooth functions defined on $[a,b]$ and $[a,b]\times[a,b]\times R$, respectively.

Nonlinear Volterra integral equations arise in various applications such as mathematical physics, engineering, and computer graphics[1-3], thus leading to an extensive literature on integral equations and their applications. Iterative methods have been found to be highly important for solving many problems. For example, there are many applications of iterative method has been used to solve fractional diffusion equation[4], the generalized inverse[5], the general nonlinear quasi-variational inequalities[6], the non-Hermitian singular saddle point problems[7] and the Fredholm integral equations[8,9], etc. In our study, we utilize an iterative method introduced by Borzabadi and Fard[9] as an efficient approach for obtaining numerical solution for (1). However, it should be noted that this method has limitations as discussed in Reference[9], where it requires changing and narrowing down the interval of the integral equation. In this paper, appropriate conditions on λ_p and k_p are imposed so as to get numerical solution for (1) within $[a,b]$. We assume that

(i) $k_p\in L^2(\Omega)$, where $\Omega=[a,b]\times[a,b]\times R$;

(ii) $\sum_{p=1}^{M}\gamma_p|\lambda_p|<\frac{1}{b-a}$, where $\gamma_p=\sup\left\{\left|\frac{\partial k_p}{\partial u}(x,y,u)\in\Omega\right|\right\}$, $p=1,2,\cdots,M$.

2. METHOD OF SOLUTION

Let $i=0,1,2,\cdots,N$, $x_i=a+\frac{b-a}{N}i$, $h_i=x_{i+1}-x_i$, $h=\max\{h_i\}$. Taking the collocation points x_i in (1), we have

$$u(x_i)=f(x_i)+\sum_{p=1}^{M}\lambda_p\int_{a_p}^{x_i}k_p(x_i,y,u(y))dy. \tag{2}$$

*wangkeyan2013@126.com

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Let $v_i \geq 1, i = 0,1,\cdots,N$, $w_{pj}^{(i)} = \int_{a_p}^{x_i} \ell_j(y)dy$ and $O(h^{v_i})$ are the integral coefficient and the truncation error, respectively. Using the interpolation-type quadrature formula, we see that

$$\lambda_p \int_{a_p}^{x_i} k_p(x_i, y, u(y))dy = \lambda_p \sum_{j=0}^{N} w_{pj}^{(i)} k_p(x_i, y_j, u(y_j)) + O(h^{v_i}). \tag{3}$$

Combine (2) and (3) to observe that

$$u(x_i) = f(x_i) + \sum_{p=1}^{M} \sum_{j=0}^{N} \lambda_p w_{pj}^{(i)} k_p(x_i, y_j, u(y_j)) + O(h^{v_m}), \quad i = 1,2,\cdots,N, \tag{4}$$

where $v_m = \min\{v_i\}$.

In order to make an analysis, we define the following system

$$\bar{u}(x_i) = f(x_i) + \sum_{p=1}^{M} \sum_{j=0}^{N} \lambda_p w_{pj}^{(i)} k_p(x_i, y_j, \bar{u}(y_j)), i = 0,1,\cdots,N. \tag{5}$$

Theorem 2.1. Let $u_i = u(x_i)$ and $\bar{u}_i = \bar{u}(x_i)$ $(i = 1,2,\cdots,N)$ are the solutions of (4) and (5), respectively. Assuming that (i) and (ii) hold, then

$$|u_m - \bar{u}_m| \leq \frac{O(h^{v_m})}{1 - (b-a)\sum_{p=1}^{M} \gamma_p |\lambda_p|} \tag{6}$$

where $|u_m - \bar{u}_m| = \max_{0 \leq i \leq N} |u_i - \bar{u}_i|$.

Proof. Set $u_j = u(y_j)$ and $\bar{u}_j = \bar{u}(y_j)$. Subtracting (5) from (4), we obtain

$$u_m - \bar{u}_m = \sum_{p=1}^{M} \sum_{j=0}^{N} \lambda_p w_{pj}^{(i)} \left[k_p(x_i, y_j, u(y_j)) - k_p(x_i, y_j, \bar{u}(y_j))\right] + O(h^{v_m}) \tag{7}$$

where $0 \leq m \leq N$. Using the conditions (i) and (ii), we see that

$$\left|k_p(x_m, y_j, u_j) - k_p(x_m, y_j, \bar{u}_j)\right| = \left|\frac{\partial k_p}{\partial u}(x_m, y_j, \xi_j)(u_j - \bar{u}_j)\right| \leq \gamma_p (u_j - \bar{u}_j), \tag{8}$$

where ξ_j is located between u_j and $\bar{u}_j$. Now, we can find an estimate for (7) using (8) to get

$$|u_m - \bar{u}_m| \leq \sum_{p=1}^{M} \sum_{j=0}^{N} |\lambda_p| \cdot w_{pj}^{(i)} \cdot \gamma_p \cdot |(u_j - \bar{u}_j)| + |O(h^{v_m})| \leq (b-a) \cdot |u_m - \bar{u}_m| \cdot \sum_{p=1}^{M} \gamma_p |\lambda_p| + |O(h^{v_m})|. \tag{9}$$

From (9), we can derive (6).

Corollary 2.1. If $N \to \infty$ in Theorem 2.1 (i.e., $h \to 0$), then $|u_m - \bar{u}_m|$ of (6) vanishes as h approaches zero.

Consequently, we can get a numerical solution for (6). Subsequently, Newton's iterative method is employed to compute an approximate solution for this nonlinear system (6), which follows a similar procedure used for solving linear systems.

Hereby, we construct a sequence of vector $\{\bar{u}^{(k)}\}$ during the iterative process and use the following iterative formula:

$$\bar{u}^{(k+1)}(x_i) = f(x_i) + \sum_{p=1}^{M} \sum_{j=0}^{N} \lambda_p w_{pj}^{(i)} k_p(x_i, y_j, \bar{u}^{(k)}(y_j)), \tag{10}$$

where $i=1,2,\cdots,N$ and $k=0,1,\cdots$. The iterative process for solving (10) as follows:

Step 1. We take $\varepsilon>0$ along with an initialilization vector $\overline{u}^{(0)}$ =0 and construct system

$$h_i=\overline{u}^{(k+1)}(x_i)-f(x_i)-\sum_{p=1}^{M}\sum_{j=0}^{N}\lambda_p w_{pj}^{(i)}k_p(x_i,y_j,\overline{u}^{(k)}(y_j)),\ i=0,1,\cdots,N. \tag{11}$$

Step 2. We calculate the Jacobian matrix $F'=\left[\frac{\partial h_i}{\partial x_i}\right]$, which is an $(N+1)\times(N+1)$ matrix.

Step 3. We calculate $\overline{u}_i^{(k+1)}$ using Step 1.

Step 4. We calculate $\left\|\overline{u}_i^{(k+1)}-\overline{u}_i\right\|_\infty$, if $\left\|\overline{u}_i^{(k+1)}-\overline{u}_i\right\|_\infty<\varepsilon$, stop; Else, take $k=k+1$ and goto Step 3.

Thus, according to (11), we can obtain iterative approximate solutions $\overline{u}_i^{(k)}=\overline{u}^{(k)}(x_i)\ (i=0,1,\cdots,N)$ for equation (1).

3. CONVERGENCE ANALYSIS

Based on the iteration method in Section 2, it is expected that as $k\to\infty$, $\overline{u}^{(k)}(x_i)$ of (10) will approximate to $\overline{u}(x_i)$ of (5). In the following, we start to prove the convergence result of the iteration solution.

Theorem 3.1. Let $\{\overline{u}^{(k)}\}$ and $\overline{u}(x_i)$ are the solutions of (10) and (5), respectively. If the conditions (i) and (ii) hold, then we have

$$\lim_{k\to\infty}\left\|\overline{u}^k-\overline{u}\right\|_\infty=0, \tag{12}$$

where $\overline{u}^{(k)}=(\overline{u}_0^{(k)},\overline{u}_1^{(k)},\cdots,\overline{u}_N^{(k)}), \overline{u}=(\overline{u}_0,\overline{u}_1,\cdots,\overline{u}_N)$.

Proof. Set $\overline{u}_j^{(k)}=\overline{u}^{(k)}(y_j)$ and $\overline{u}_j=\overline{u}(y_j)$. It follows from (5) and (10) that

$$\overline{u}_i^{(k+1)}-\overline{u}_i=\sum_{p=1}^{M}\sum_{j=0}^{M}\lambda_p w_{pj}^{(i)}\left[k_p(x_i,y_j,\overline{u}_j^{(k)})-k_p(x_i,y_j,\overline{u}_j)\right], \tag{13}$$

Using the conditions (i) and (ii), we find that

$$\left|k_p(x_i,y_j,\overline{u}_j^{(k)})-k_p(x_i,y_j,\overline{u}_j)\right|=\left|\frac{\partial k_p}{\partial u}(x_i,y_j,\xi_j^{(k)})(\overline{u}_j^{(k)}-\overline{u}_j)\right|\le\gamma_p(\overline{u}_j^{(k)}-\overline{u}_j), \tag{14}$$

where $\xi_j^{(k)}$ is located between $\overline{u}_j^{(k)}$ and $\overline{u}_j$. From (13) and (14), we have

$$\begin{aligned}\left\|\overline{u}_i^{(k+1)}-\overline{u}_i\right\|_\infty&\le\sum_{p=1}^{M}\sum_{j=0}^{N}\left|\lambda_p\right|\cdot w_{pj}^{(i)}\cdot\gamma_p\cdot\left|\overline{u}_i^{(k)}-\overline{u}_j\right|\\&\le(b-a)\cdot\left\|\overline{u}_i^{(k)}-\overline{u}_i\right\|_\infty\cdot\sum_{p=1}^{M}\gamma_p\left|\lambda_p\right|\\&\le[(b-a)\cdot\sum_{p=1}^{M}\gamma_p\left|\lambda_p\right|]^2\cdot\left\|\overline{u}_i^{(k-1)}-\overline{u}_i\right\|_\infty\\&\le\cdots\le[(b-a)\cdot\sum_{p=1}^{M}\gamma_p\left|\lambda_p\right|]^k\cdot\left\|\overline{u}_i^{(0)}-\overline{u}_i\right\|_\infty.\end{aligned}$$

Noting that $\sum_{p=1}^{M}\gamma_p|\lambda_p| < \frac{1}{b-a}$, when $k \to \infty$, we conclude that $\lim_{k\to\infty}\left\|\bar{u}_i^{(k+1)} - \bar{u}_i\right\|_\infty = 0$, which shows that (12) holds.

4. NUMERICAL EXAMPLE

In this section, a numerical example is provided to support the theoretical analysis using Matlab R2012b on a personal computer. Let $e(x_i) = \left|u(x_i) - \bar{u}^{(k)}(x_i)\right|,\ i = 1,2,\cdots,N,\ k = 0,1,\cdots$.

Example 4.1. Consider the following nonlinear Volterra integral equation with piecewise intervals

$$u(x) = \frac{e^x}{4000}\left[4000 + e^2 + e^x\left(20x - 40x^2 - e^{3x}\right)\right] - \frac{e^{\frac{4}{5}}x}{1000} + \frac{1}{50}\int_{\frac{2}{5}}^{x} xyu^2(y)dy + \frac{1}{50}\int_{\frac{1}{2}}^{x} e^{x+y}u^3(y)dy,\ \ 0 \le x \le 1,$$

with the exact solution $u(x) = e^x$. In this experiment, the conditions (i) and (ii) are considered. Choosing $\varepsilon = 10^{-10}$ and $\bar{u}^{(0)} = 0$. Figure 1 displays a comparison between $u(x_i)$ and $\bar{u}^{(k)}(x_i)$, while Figure 1 also displays the absolute error function $e(x_i)$ for $N = 10$. The iterative calculation process and results are presented in Table 1 which includes calculated iterative solutions and absolute errors obtained using iterative method for $N = 10$. From Figure 1 and Table 1, we see that the iterative method is very effective in numerical solution of nonlinear Volterra integral equation with piecewise intervals.

Table 1. Numerical results of iterative method.

k	$\bar{u}^{(k)}(1/10)$	$\bar{u}^{(k)}(3/10)$	$\bar{u}^{(k)}(7/10)$	$\bar{u}^{(k)}(9/10)$	$e(1/10)$	$e(3/10)$	$e(7/10)$	$e(9/10)$
1	1.107066	1.351658	2.001959	2.417861	1.89e-3	1.79e-3	1.17e-2	4.17e-2
2	1.104700	1.349857	2.015500	2.461003	4.71e-4	1.81e-6	1.74e-3	1.39e-3
3	1.104703	1.349852	2.013771	2.459712	4.67e-4	6.81e-6	1.82e-5	1.08e-4
4	1.104704	1.349853	2.013751	2.459601	4.66e-4	5.81e-6	1.71e-6	2.11e-6
5	1.104704	1.349853	2.013754	2.459603	4.66e-4	5.81e-6	1.29e-6	1.11e-7
6	1.104704	1.349853	2.013754	2.459604	4.66e-4	5.81e-6	1.29e-6	8.88e-7

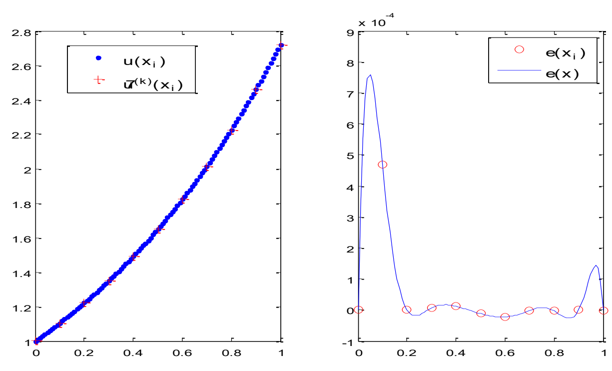

Figure 1. The exact solution $u(x_i)$, the approximate solution $\bar{u}^{(k)}(x_i)$ and the error function $e(x_i)$ of Example 4.1.

5. CONCLUSION

In this article, we have discussed the convergence analysis of iterative method for nonlinear Volterra integral equation with piecewise intervals. Our numerical results confirm our analysis and show that the proposed scheme is efficient. In the near future, we shall consider iterative method for two-dimensional nonlinear Volterra integral equation with piecewise intervals and more complicated problems.

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