The role of Fourier theory in optics education

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ABSTRACT

Fourier theory has found application in many areas of science and engineering, but perhaps nowhere has it found a more natural home than in the field of optics. Why? Because optics, by its very nature, allows the utility of this incredible mathematical tool to be viewed directly; in other disciplines, its consequences can usually only be imagined or, at best, observed indirectly. Such a unique characteristic serves as a powerful educational aid for those in the process of learning about the field. The thesis that a polychromatic optical wavefield consists of a superposition of temporal-frequency components is easily demonstrated in the classroom, as is the decomposition of an arbitrary monochromatic wavefield into its plane-wave spectrum. Yet another demonstration helps explain the spatial-frequency performance of an optical imaging system. The now-popular Fourier techniques that provide a solid mathematical foundation for exploring the physics associated with these and other phenomena are discussed.

1. INTRODUCTION

The Fourier transform is an extremely important mathematical tool in many areas of science and engineering, particularly those areas involving phenomena that can be represented by functions composed of sine and cosine functions; e.g., signal theory, vibration analysis, acoustics, turbulence theory, electromagnetic wave theory, etc. In each of these areas, the elegance and power of Fourier techniques can often greatly simplify the mathematics involved in solving problems or analyzing systems, but perhaps only in the field of optics can the consequences of these techniques be observed directly. To illustrate, the temporal-frequency spectral composition of a polychromatic optical wavefield can be displayed for direct viewing by passing it through a dispersive medium, such as a prism, which separates the various components according to color (i.e., wavelength). Also, the spatial-frequency spectral composition of a monochromatic optical wavefield can be observed visually by viewing the Fraunhofer diffraction pattern of that wavefield. As a result, optics provides a natural home for the use of Fourier techniques. In contrast, while Fourier theory remains an extremely powerful tool in areas other than optics, its consequences can normally only be imagined or observed indirectly. Here we shall concentrate on the role of this beautiful mathematical theory in optics education.

2. MATHEMATICAL BACKGROUND

Because this subject is covered in great detail elsewhere[1-4], we will include here only those parts of the mathematical background that are essential for this brief effort.

Even though it was nearly 200 years ago that Jacques Fourier undertook a systematic study of the series and integral expansions that now bear his name, applications of his work have perhaps never found more widespread use than they do today. As is well known, it is often more convenient to work with the frequency-domain representation of a function instead of the original function itself. This frequency-domain representation is associated with a decomposition of the function into a collection of elementary functions which, in the case of the Fourier decomposition, are sine and cosine functions. We assume, of course, that any functions we shall encounter are either well behaved or pose no insurmountable mathematical difficulties (e.g., functions that represent observable physical phenomena, that possess Fourier transforms in-the-limit, etc.). Let us consider the Fourier representation of such a function, $f(x)$:

$$f(x) = \int_{-\infty}^{\infty} F(\alpha)e^{j2\pi\alpha x}\,d\alpha \ .$$

(1)

If we denote by ξ the frequency variable associated with x, the function $F(\xi)$ is seen to behave as a frequency-dependent weighting factor. Thus, $f(x)$ may be considered to be composed of a collection of complex exponentials (sines and cosines),

each of which is weighted in amplitude and phase according to its frequency. This weighting factor, which is also known as the Fourier transform of f(x), is given by

$$F(\xi) = \int_{-\infty}^{\infty} f(\beta)e^{-j2\pi\beta\xi} \, d\beta \, , \tag{2}$$

and the functions f(x) and $F(\xi)$ are said to form a Fourier transform pair. We now list a few common Fourier transform pairs:

f(x)	F(ξ)	
$\delta(x)$	1	(3)
$\cos(2\pi\xi_0 x)$	$\frac{1}{2}\left[\delta(\xi+\xi_0)+\delta(\xi-\xi_0)\right]$	(4)
rect(x)	$\text{sinc}(\xi)$	(5)
$\exp(-\pi x^2)$	$\exp(-\pi\xi^2)$	(6)
comb(x)	$\text{comb}(\xi)$.	(7)

Here ξ_0 is a real-valued constant, $\delta(x)$ is the Dirac delta function, and the rectangle, sinc and comb functions are defined by

$$\begin{aligned} \text{rect}(x) &= 1, \quad -.5 \le x \le .5 \\ &= 0, \quad \text{otherwise} \end{aligned} \tag{8}$$

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \tag{9}$$

$$\text{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x-nx_0). \tag{10}$$

Let us assume that $F(\xi)$, $H(\xi)$ and $G(\xi)$ are the Fourier transforms of f(x), h(x) and g(x), respectively, and let us denote two additional real-valued constants by the symbols x_0 and d and two complex-valued constants by the symbols a and b. We next introduce a number of extremely powerful properties and theorems associated with the Fourier transform:

Property/Theorem	g(x)	G(ξ)	
Linearity	a f(x) + b h(x)	a F(ξ) + b H(ξ)	(11)
Scaling	$f(\frac{x}{d})$	$\lvert d\rvert F(d\xi)$	(12)
Shifting	$f(x-x_0)$	$F(\xi)e^{-j2\pi x_0\xi}$	(13)
Transform of F(x)	F(x)	$f(-\xi)$	(14)
Convolution	$\int_{-\infty}^{\infty} f(\alpha)h(x-\alpha)d\alpha$	$F(\xi)H(\xi)$	(15)
Central ordinate	$f(0) = \int_{-\infty}^{\infty} F(\alpha)d\alpha.$		(16)

We have limited our presentation of the basic mathematical tools of Fourier theory to one-dimensional functions for brevity, and by now extending these notions to two dimensional functions we will be able to explore a number of applications in optics education: specifically, diffraction and image formation.

3. APPLICATIONS IN DIFFRACTION

To begin our discussion of Fourier applications in diffraction, let us consider the situation in which a diffracting aperture is illuminated by a monochromatic wavefield as depicted in Fig.1. The amplitude of the transmitted wavefield is given by the product

$$u_1^+(x,y) = u_1^-(x,y) \, t_1(x,y), \tag{17}$$

where $u_1^-(x,y)$ is the amplitude of the incident wavefield, $t_1(x,y)$ is the amplitude transmittance of the aperture, and the positive direction for the y coordinate is out of the page. Our ultimate goal is to describe the irradiance of the transmitted wavefield after it has propagated to the observation plane located a distance z_{12} away.
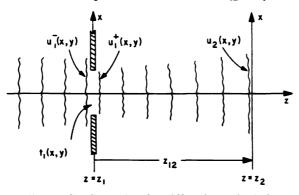

Fig. 1. Configuration for diffraction calculations.

If we now simplify matters by requiring the illuminating wavefield to be a normally incident planewave of amplitude A, the transmitted wavefield may be written as the product of A and $t_1(x,y)$. If we further require that the propagation distance be large enough to place the observation plane in the Fraunhofer region of the aperture, the observed wavefield amplitude will be proportional to the Fourier transform of the transmitted wavefield; i.e.,

$$u_2(x,y) \propto A \, T_1\left(\frac{x}{\lambda z_{12}}, \frac{y}{\lambda z_{12}}\right), \tag{18}$$

where λ is the wavelength of the light. Finally, the irradiance of the Fraunhofer diffraction pattern, which is proportional to the squared modulus of the wavefield amplitude, may be expressed as

$$E_2(x,y) = \left| \frac{A}{\lambda z_{12}} \, T_1\left(\frac{x}{\lambda z_{12}}, \frac{y}{\lambda z_{12}}\right) \right|^2. \tag{19}$$

To illustrate a specific example, let us calculate the irradiance of the Fraunhofer diffraction pattern of a rectangular aperture of height c in the x-direction and width d in the y-direction. For such an aperture we have

$$t_1(x,y) = \text{rect}\left(\frac{x}{c}, \frac{y}{d}\right), \tag{20}$$

$$E_2(x,y) = \left| \frac{Acd}{\lambda z_{12}} \right|^2 \left| \text{sinc}\left(\frac{cx}{\lambda z_{12}}, \frac{dy}{\lambda z_{12}}\right) \right|^2. \tag{21}$$

This is a mathematical description of the familiar multi-lobed distribution associated with the Fraunhofer diffraction pattern of a rectangular aperture. Because this irradiance distribution is proportional to the squared modulus of the Fourier transform of the aperture function, the properties and theorems of Fourier transforms listed above may be used to explore various relationships between these two quantities. For example, if the aperture is allowed to become wider in either the x or y direction, or both, the diffraction pattern becomes correspondingly narrower in the same direction(s) -- a result that follows from the scaling property of Fourier transforms. Note that Eq. (21) was obtained by using two-dimensional versions of Eqs.

(5) and (12). It is interesting to observe that the magnitude of the diffraction pattern irradiance is proportional to the square of the area of the aperture, a result that holds for any clear aperture illuminated by a uniform wavefield. Why the square of the aperture area and not simply the area? This may be explained as follows: the power transmitted by the aperture is proportional to the area of the aperture, but the associated Fraunhofer diffraction pattern is spread over a region whose two-dimensional extent is inversely proportional to the area of the aperture; consequently, the irradiance of the Fraunhofer diffraction pattern varies as the square of the aperture area.

There may be occasions when only the value of the irradiance at the center of the Fraunhofer pattern is required, and not the behavior of the entire pattern. That value may be determined, of course, by simply evaluating the expression for the irradiance at the origin -- the nominal "center" of the pattern. Doing so for the case of the uniformly illuminated rectangular aperture, we obtain

$$E_2(0,0) = | \frac{Acd}{\lambda z_{12}} |^2 . \tag{22}$$

Thus, the irradiance at the origin is proportional to the square of the area of the aperture and depends on the aperture in no other way. However, the same result could have been determined directly by using a technique based on the central-ordinate theorem, thereby eliminating the need to calculate the behavior of the entire diffraction pattern. This technique combines two-dimensional versions of Eqs. (16) and (14) and leads to the general result

$$E_2(0,0) = | \frac{1}{\lambda z_{12}} |^2 | \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_1^+(\alpha,\beta)d\alpha d\beta |^2 , \tag{23}$$

which reduces to Eq. (22) for the case of the clear rectangular aperture described above. This method may also be used to find the irradiance at the origin when the aperture is not a clear aperture, when it is not illuminated by a uniform wavefield (or both), or anytime the integral of the transmitted wavefield is more easily calculated than its Fourier transform.

It should be noted that Fraunhofer diffraction patterns may be observed much closer to diffracting objects than implied by the statements above. For example, if the object is illuminated by a spherically converging wave rather than a plane wave, as shown in Fig. 2, the associated Fraunhofer pattern is found in the plane to which the spherical wave is converging. The pattern produced in this fashion will exhibit the same form as before, but its scale will be smaller because the distance from the object to the Fraunhofer plane is smaller. In addition, even when plane-wave illumination is used, the Fraunhofer pattern can be observed at the back focal plane of a lens placed after the object, as shown in Fig. 3.
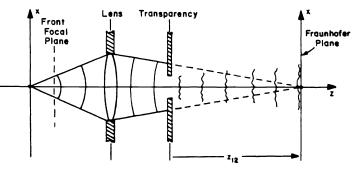

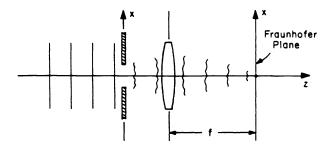Fig. 2. Diffraction with converging spherical wave illumination.

Fig. 3. Fraunhofer pattern located at back focal plane of the lens.

Again the form of the pattern remains the same, but the scaling distance in the argument of the diffraction pattern expressions is now the focal length of the lens, f, rather than z_{12} as before. Of course, the Fourier relationships that were noted previously between object and Fraunhofer pattern widths, object area and the central irradiance of the pattern, etc., still hold. The effects of lenses on the behavior of diffraction is made use of in the discussion of image formation that follows.

4. APPLICATIONS IN IMAGE FORMATION

We now investigate the formation of images, first with coherent light and a bit later with incoherent light. Let us assume that we wish to form an image of the object shown in Fig. 1. It is instructive to describe the transmitted wavefield in terms of its Fourier components, i.e.,

$$u_1^+(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_1^+(\xi',\eta')e^{j2\pi(\xi'x + \eta'y)}d\xi'd\eta' ,$$

(24)

where $U_1^+(\xi,\eta)$ is the Fourier transform of $u_1^+(x,y)$. If we now set $\xi=\gamma_x/\lambda$, $\eta=\gamma_y/\lambda$ and $k = 2\pi/\lambda$, we see that this Fourier decomposition also represents a decomposition into components of the form

$$u(x,y) = \exp[jk(\gamma_x x + \gamma_y y)] .$$

(25)

This expression represents a planewave propagating with direction cosines γ_x and γ_y relative to the x and y axes, respectively, and k is the associated wavenumber. Thus, the wavefield of Eq. (24) may be thought of as consisting of a superposition of planewave components, each weighted in amplitude and phase according to its direction of propagation. Although not physically possible, if an identical combination of planewaves could be reproduced at some other location (i.e.,with exactly the same weighting of amplitude and phase as a function of propagation direction), a "perfect" image would result. However, if a collection of planewaves with modified weighting is formed, as is the case for a real-world imaging system, a less-than-perfect image will result.

Let us now consider the two-element imaging system shown in Fig. 4. The quality of an image produced by such a system is related to a number of characteristics, but perhaps central in importance is the aperture stop. When imaging with coherent light, the first lens element of the system causes the Fraunhofer diffraction pattern of the object to impinge upon the plane of the aperture stop, which then allows only a portion of the incident wavefield to be transmitted. The remaining lens element subsequently forms a Fraunhofer diffraction pattern of this transmitted light at the location of the image plane. Because the Fraunhofer pattern incident on the aperture stop represents a frequency-domain description of the object wavefield, because the wavefield transmitted by the stop represents the frequency-domain description of the image wavefield, and because the amplitude of the wavefield transmitted by the stop is described as a product of the incident wavefield amplitude and the amplitude transmittance of the stop, the stop effectively performs the role of a filter.
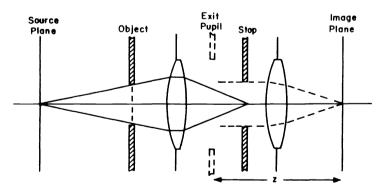

Fig. 4. Configuration for image formation with coherent light.

To simplify analysis problems and simultaneously account for aberrations, a system is often modeled by its exit pupil function, which is an image of the aperture stop -- viewed from image space -- that has been modified to include aberration effects. Thus, in the language of linear systems, the exit pupil effectively serves as a transfer function by altering the magnitude and phase of the various Fourier (planewave) components associated with the object as these components pass through the system. To avoid confusion with notation, let us choose an overall magnification of unity and an image distance

of z (i.e., the distance from the exit pupil to the image plane is equal to z). Then, with the object wavefield denoted by $u_o(x,y)$, the image wavefield by $u_i(x,y)$ and the exit pupil function by $p(x,y)$, we find that

$$U_i(\xi,\eta) = KH(\xi,\eta)U_o(\xi,\eta) , \qquad (26)$$

where $U_i(\xi,\eta)$ and $U_o(\xi,\eta)$ are the spectra of the image and object, respectively, K is a constant, and

$$H(\xi,\eta) = p(-\lambda z\xi,-\lambda z\eta) \qquad (27)$$

is the system transfer function. The wavefield amplitude of the image is then found by performing an inverse Fourier transform of the image spectrum, i.e.,

$$u_i(x,y) = K \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(-\lambda z\xi,-\lambda z\eta)U_o(\xi',\eta')e^{j2\pi(\xi'x + \eta'y)}d\xi'd\eta' . \qquad (28)$$

Note that this expression represents a linear mapping of object wavefield amplitude into image wavefield amplitude. To illustrate with a specific example, let us choose an unaberrated system with a circular exit pupil of diameter d. Thus, the transfer function may be written

$$H(\xi,\eta) = rect\left(\frac{\lambda z}{d}\sqrt{\xi^2+\eta^2}\right), \qquad (29)$$

the cutoff frequency is equal to $|d/\lambda z|$, and the amplitude of the image wavefield for this aperture will contain only those Fourier components whose frequencies lie below this value. The cutoff frequency, and thus the image quality, is seen to depend directly on the value of the aperture stop diameter: the image quality improves when the stop is made larger, and vice versa (assuming no aberrations, of course). The coherent impulse response, which is simply the Fourier transform of the transfer function, has the form

$$h(x,y) \propto \frac{J_1\left(\frac{\pi d}{\lambda z}\sqrt{x^2+y^2}\right)}{\frac{\pi d}{\lambda z}\sqrt{x^2+y^2}} , \qquad (30)$$

where $J_1(r)$ is the first-order Bessel function of the first kind. The so-called resolution of this imaging system is related to the width of its impulse response, which can be seen to depend inversely on the diameter of the aperture. The squared modulus of this distribution describes the well-known Airy pattern.

We now finish our discussion regarding the role of Fourier theory in optics education with a brief treatment of image formation when incoherent light of reasonably narrow spectral bandwidth is used. We shall use the configuration of Fig. 5.
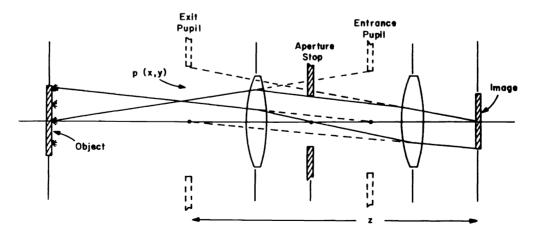

Fig. 5. Configuration for image formation with incoherent light.

Although there are similarities between coherent and incoherent image formation, there are substantial differences. For example, instead of a linear mapping of wavefield amplitude to wavefield amplitude, we find a linear mapping of object radiance into image irradiance. Thus, if $L_o(x,y)$ denotes the object radiance distribution, then the image irradiance is given by

$$E_i(x,y) = K \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{H}(\xi',\eta') \mathcal{L}_o(\xi',\eta') e^{j2\pi(\xi'x + \eta'y)} d\xi' d\eta' , \tag{31}$$

where $\mathcal{L}_o(\xi,\eta)$ is the Fourier transform of $L_o(x,y)$ and $\mathcal{H}(\xi,\eta)$ is the optical transfer function (OTF). Because the notation becomes quite cumbersome, we shall not attempt to express the OTF explicitly, but will merely state that it is proportional to the complex autocorrelation of the exit pupil function, i.e.,

$$\mathcal{H}(\xi,\eta) \propto \text{Complex autocorrelation of } p(-\lambda z\xi, -\lambda z\eta). \tag{32}$$

Using various properties of Fourier transforms, it may be shown that the incoherent impulse response, or point spread function (PSF), is proportional to the squared modulus of the Fourier transform of the exit pupil function. In other words,

$$h(x,y) \propto \left| P\left(\frac{x}{\lambda z}, \frac{y}{\lambda z}\right) \right|^2 , \tag{33}$$

where $P(\xi,\eta)$ is the Fourier transform of $p(x,y)$. For the case of the circular exit pupil of diameter d, this becomes

$$h(x,y) \propto \left| \frac{J_1\left(\frac{\pi d}{\lambda z} \sqrt{x^2+y^2}\right)}{\frac{\pi d}{\lambda z} \sqrt{x^2+y^2}} \right|^2 , \tag{34}$$

which, as mentioned above, describes the Airy pattern. In conclusion, we again note that the system resolution is inversely proportional to the diameter of the exit pupil, from which it follows that the cutoff frequency depends directly on the diameter of the exit pupil.

5. AKNOWLEDGEMENTS

6. REFERENCES

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